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Department of AERONAUTICS and ASTRONAUTICS
STANFORD UNIVERSITY

NICHOLAS J. HOFF

DYNAMIC STABILITY OF STRUCTURES

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Department of Aeronautics and Astronautics
Stanford University
Stanford, California

DYNAMIC STABILITY OF STRUCTURES

by

Nicholas J. Hoff

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Summary

Dynamic stability is defined and classified, and examples are given for the various classes of problems. Criteria are developed for practical stability and it is shown that in a practical elastic column tested in a conventional testing machine stress reversal always precedes the attainment of the maximum load. The two coincide, however, in the limit when the initial deviations of the column axis from straightness and the loading speed tend to zero.

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Introduction

Thirty-five years ago, when he was working in the airplane industry, the author became aware of the importance of dynamic stability. As a beginning engineer, he had to carry out the complete dynamic, aerodynamic, and structural analysis of a new training plane which was then being designed by four of his equally inexperienced colleagues. It may be of interest to add that in one year's time the design and analysis were completed and the prototype manufactured and successfully test flown; the story of the difficulties encountered is not part of the present paper.

In accordance with governmental airworthiness requirements, the static stability of the straight-line flight of the plane had to be checked under climbing, horizontal flying and gliding conditions. This was done by plotting first against the angle of attack the aerodynamic wing moments with respect to an axis through the center of gravity of the airplane perpendicular to the plane of symmetry of the airplane. The curve obtained always indicates instability for the conventional airplane: If the angle of attack α of the wing is increased, the aerodynamic wing moment M also increases (See Fig. 1).*

The airplane is stabilized with the aid of the horizontal tail surfaces. The change in the moment of the tail surface with respect to its own centroidal axis is insignificant compared to the change in the moment of the vertical upward force acting on the tail surface with

*Some of the aerodynamic and inertia forces necessary for dynamic equilibrium are not shown in the figure.

respect to the center of gravity of the airplane. An increase in the angle of attack α of the wing causes an increase in the angle of attack of the tail surface which increases the lifting force P_H of the horizontal tail surface. This increment ΔP_H multiplied with the distance h between the center of gravity of the airplane and the line of action of the tail force is a negative moment capable of counterbalancing the positive increment ΔM in the wing moment if h and the area of the tail surface are sufficiently large. The job of the designer-analyst was therefore to insure, under all normal flight conditions, that $h(dP_H/d\alpha)d\alpha$ had a greater absolute value than $(dM/d\alpha)d\alpha$.

It was quite a revelation for the author to find out from the literature[1] that the condition mentioned was only a necessary but not a sufficient condition of an automatically stable flight of the airplane. If the dynamic equations of the motion of a rigid airplane in its own plane of symmetry are written for small oscillations about the steady-state conditions, and if the aerodynamic coefficients are inserted with their steady-state values, assumption of an exponential solution results in a quartic. Solution of the quartic can lead to any of the four different behavior patterns shown in Fig. 2. Curve a indicates an asymptotic return to the initial state after a disturbance, which obviously means stability of the initial state. A second stable pattern is shown in curve b where the disturbance is followed by damped oscillations and eventually a return to the initial state. Curve c corresponds to static and dynamic instability. Finally, the airplane characterized by curve d is statically stable but dynamically unstable; the static restoring force or moment acts in the sense necessary for stability but the ensuing oscillations increase in amplitude rather than damp out.

In the literature of structural stability one can also find systems that appear to be fully stable when investigated by static methods and whose displacements from the state of initial equilibrium nevertheless increase with time following a disturbance, just as it is indicated in Fig. 2d. Such systems evidently must be analyzed with the aid of the dynamic, or kinetic, method in which the motion of the system following a disturbance is studied. The dynamic equations of motion are also needed when the loads applied to the structure vary significantly with time.

In a recent paper Herrmann and Bungay[2] followed usage established in the theory of aeroelasticity when they proposed that structural instability of type (c) be designated as divergence, and that of type (d) as flutter.

Definition and classification of problems of dynamic stability

Definition

In the introduction to the first English edition of their monumental textbook entitled Engineering Dynamics, Biezeno and Grammel[3] explain that, following Kirchhoff's definition, dynamics is the science of motion and forces, and thus includes statics, which is the study of equilibrium, and kinetics, which treats of the relationship between forces and motion. According to this interpretation of the meaning of the words, the dynamic test of equilibrium mentioned in the Introduction of this paper should be called a kinetic test, and this is indeed the terminology adopted by Ziegler[4] in his studies of the stability of non-conservative systems. But dynamics is generally accepted as the

antonym of statics in everyday usage, and this is the sense in which it is used in the title of the International Conference on Dynamic Stability of Structures.

A number of significantly different concepts can be included in the meaning of the term dynamic stability of structures. One of them is the stability of motion of an elastic system subjected to forces that are functions of time. Another is the study of the stability of a system subjected to constant forces as long as the study is carried out with the aid of the dynamic equations of motion; such an investigation is designated by Ziegler as a stability analysis with the aid of the kinetic criterion.

In this paper any stability problem analyzed with the aid of Newton's equations of motion, or by any equivalent method, will be considered a dynamic stability problem.

The classification of the problems that follow is not fundamental in any sense of the word. Its purpose is simply to group together problems that are usually treated by similar mathematical methods, or analyzed by the same group of research men. A more fundamental classification could be based on the principles proposed by Ziegler[5] in his article in Advances in Applied Mechanics.

Parametric resonance

Among the problems of the dynamic stability of structures probably the best known subclass is constituted by the problems of parametric excitation, or parametric resonance. A typical example is the initially straight prismatic column whose two ends are simply

supported and upon which a periodic axial compressive load is acting (Fig. 3). Such a column is known to develop lateral oscillations if its straight-line equilibrium is disturbed. Depending upon the magnitude and the frequency of the pulsating axial load, the linear Hill or Mathieu equation defining the lateral displacements of the column may yield bounded or unbounded values for these displacements. The structural analyst can be useful to the design engineer if he points out the regions in the frequency-amplitude plane that must be avoided if the column should never deviate noticeably from its initial straight-line equilibrium configuration.

According to Bolotin[6], the first solution of this problem was given by Beliaev[7] in 1924; this was followed by an analysis by Krylov and Bogoliubov[8] in 1935. In the United States, Lubkin[9], a student of Stoker, solved the problem in a doctoral dissertation submitted to New York University in 1939; the results are more easily available in an article by Lubkin and Stoker[10] printed in 1943. The results of a theoretical and experimental investigation of the subject were published by Utida and Sezawa[11] in 1940. Another early solution of the parametric excitation problem of the column is due to Mettler[12] (1940); as a matter of fact, this problem is called Mettler's problem in Ziegler's comprehensive work on elastic stability. The parametric excitation of thin flat plates was first discussed by Einaudi[13] (1936). A rather complete treatment of known solutions of the parametric resonance problem can be found in a book by Bolotin[6] in which the effects of friction and nonlinearities are also discussed.

It is perhaps unfortunate that the phenomena described have become known as parametric excitation, or parametric resonance. This terminology refers to the mathematical structure of the equations defining the phenomena, and fails to define or allude to the physical nature of the phenomena. Perhaps the terms sympathetic excitation or sympathetic resonance would be more descriptive and acceptable.

It is worth noting that in his book already mentioned, Bolotin[6] defines dynamic buckling much more restrictively than was done in this paper. He includes only the phenomena of parametric excitation and resonance in his definition.

Two papers to be presented at this conference deal, at least in part, with parametric resonance, namely those by E. Mettler, and by D. A. Evensen and R. E. Fulton. Possibly the presentations by V. V. Bolotin and by S. T. Ariaratnam, which treat of stability under random loading, will also make use of the techniques employed in parametric resonance studies.

Impulsive loading

In the second subclass of the dynamic stability of structures, buckling under step loading and impulsive loading are studied. Because of the difficulties of tracing the origins of these studies, only a few random examples of solutions will be given here. Two early papers dealing with the danger of failure of a column subjected suddenly to a constant axial compressive load which is suddenly removed after a finite time interval of action were published by Koning and Taub[14], and by Taub[15], respectively, in 1933. Their results showed that a

suddenly applied load can cause collapse even if it is smaller than the Euler load. At the same time, the column need not be damaged by a suddenly applied load greater than the Euler load if the load is removed after a sufficiently short time.

The author of the present article, together with Victor Bruce[16], prepared a paper for the Eighth International Congress of Applied Mechanics held in Istanbul in 1952 in which the snap-through buckling of a laterally loaded perfectly elastic flat arch (see Fig. 4) was studied under step loading and impulsive loading. A diagram showing some of the results obtained for a particular case is reproduced in Fig. 5.

The figure contains the equipotential lines of a system consisting of a flat arch and a suddenly applied lateral load Q that is distributed according to a half sine wave along the arch. The initial rise $y_{o\ max} = e\rho$ of the arch of the example is such that $e = 8$; and ρ is the radius of gyration of the cross section of the arch. The magnitude of the lateral load was so selected as to be critical, that is just sufficient to cause snap-through buckling, if the load is suddenly applied and then maintained constant for all times t. Along each equipotential line the strain energy stored in the arch less the work done by the applied load is constant.

The equipotential lines can be regarded as the contour lines of a topographic map of the potential energy surface. The abscissa $r_1 = y_{1\ max}/\rho$ is the non-dimensional amplitude of the displacements according to the first symmetric mode (one half-sine wave), and $r_2 = y_{2\ max}/\rho$ is the corresponding quantity according to the first antisymmetric mode (two half-sine waves).

The undisplaced position corresponds to $r_2 = 0$ and $r_1 = e = 8$. To the left of this position is a shallow hollow; at its bottom the potential energy surface has a minimum which corresponds to stable equilibrium near the unloaded position of the arch. Farther left, at $r_1 = 1$, $r_2 = 0$ there is a hilltop where the equilibrium is unstable, and at about $r_1 = -9$, $r_2 = 0$ there is a much deeper hollow; this point of the map corresponds to stable equilibrium in a position beyond the straight line connecting the end points of the arch. If the load is large enough, the arch can snap through into this position.

The surface has no maxima or minima except along the axis of abscissae. This implies that no stable equilibrium is possible in the presence of displacements according to the antisymmetric mode. One unstable equilibrium position exists, however, near $r_1 = 5$, $r_2 = 3$ where the surface has a saddle point. The critical nature of the diagram is manifested by the fact that the initial position characterized by $r_1 = 8$, $r_2 = 0$ is connected with the saddle point by a contour line. For a slightly lower value of Q than Q_{crit} the saddle point would correspond to an energy level higher than that of the initial point; hence the arch would be unable to reach the saddle point without a finite disturbance.

With $Q = Q_{cr}$ an infinitesimal disturbance in the first antisymmetric mode can start the arch moving toward the state represented by the saddle point. If it reaches the saddle point and then continues to the left and upward along the dotted line, it is likely to descend through the ravine to the stable state of equilibrium in the snapped-through position. Along this steepest path of descent the arch gathers

speed and both $\dot{r}_1$ and $\dot{r}_2$ change rapidly in absolute value. The arch cannot stop therefore in the snapped-through state but must continue to move as its kinetic energy is now quite high. In a real system this energy will be transformed gradually into heat in consequence of friction and eventually the arch will come to rest in the snapped-through position.

It is of interest to note that in the case treated snap through can take place only with the aid of the antisymmetric mode even though the initial and final states are entirely symmetric; in an experiment the presence of the antisymmetric mode might not even be noticed because of the speed of the motion. In the absence of the antisymmetric mode, however, the energy of the arch would have to climb over the high barrier of the symmetric unstable equilibrium state which could be avoided when the arch passed through the antisymmetric mode.

The recent technical literature comprises many articles belonging to the subclass of impulsive loading. Among them may be mentioned elastic analyses of shallow shells by Grigolyuk[17] and by Humphreys and Bodner[18], and an analytical study of imperfection-sensitive elastic structures by Budiansky and Hutchinson[19]. Essentially experimental investigations of the collapse of thin-walled circular cylindrical shells under axial impact were presented by Coppa[20] and by Schwieger and Spuida[21]. Numerical methods to predict the dynamic deformations of beams, rings, plates and shells of revolution were developed by Witmer, Balmer, Leech and Pian[22]. The elastic behavior of circular cylindrical shells under lateral impact was studied by Goodier and McIvor[23] and by Lindberg[24]; the latter also gave results of experiments. Plastic deformations were also taken into account and results compared with experiment by Abrahamson and Goodier[25].

At the present conference a number of papers will be presented that belong, at least in part, to Subclass 2. Their authors are B. Budiansky; J. N. Goodier; J. M. T. Thompson; D. A. Evensen and R. E. Fulton; and T. H. H. Pian, H. Balmer, and L. L. Bucciarelli, Jr.

Circulatory loads

The third subclass is constituted by problems of buckling under stationary circulatory loads, that is, under loads not derivable from a potential and not explicitly dependent on time. The best-known example in this field is Beck's problem which is shown in Fig. 6. The problem was discovered by Pflüger[26], explained by Ziegler[27], and solved by Beck[28], a doctoral student of Ziegler. A static linear analysis leads to the conclusion that a column, one of whose ends is rigidly fixed while the other is subjected to a compressive load of constant magnitude P whose direction is always tangent to the deformed column axis, does not buckle, whatever be the magnitude of the load. On the other hand, Beck's study of the flexural vibrations disclosed that the amplitudes remain small when the initial displacements and velocities are small, provided that P is less than the critical value

$$P_{cr} = 20.05\ (EI/L^2) \tag{1}$$

where EI is the bending rigidity and L the length of the column. When $P > P_{cr}$, the amplitude of the oscillations increase without bounds. It must be concluded therefore that the column is unstable in the presence of lateral disturbances of its original state of equilibrium. It is worthy of note that P_{cr} of Eq. (1) is about eight times the Euler load of the column (which is calculated for a load P that always remains vertical).

Ziegler[4] stated in his paper already cited that non-conservative stability problems must be analyzed by means of the dynamic criterion of buckling as the two commonly used static methods of determining critical loads can lead to incorrect results. One of these methods consists of finding, by means of the static equations of equilibrium, a static equilibrium state in the immediate neighborhood of the state whose stability is being investigated. In the other method the total potential energy of the system is studied; the lowest load under which this energy ceas to be positive definite is the critical load.

In 1956 the criteria of elastic stability were analyzed at some length by Ziegler[5] who already in 1952[29] had discussed the concept of a conservative system. An additional study of stability in the presence of non-conservative forces is due to Herrmann and Bungay[2]. Finally, a recent book by Bolotin[30] is entirely devoted to the buckling problems of non-conservative systems. Bolotin attributes to Nikolai[31],[32] the discovery of the insufficiency of the static approach to the calculation of the critical load of a particular elastic system, namely a bar subjected simultaneously to compression and torsion.

At the present conference, G. Herrmann and S. Nemat-Nasser will discuss the buckling of non-conservative systems.

Aeroelastic problems

Interaction between the non-conservative aerodynamic forces and the elastic structure of airplanes and missiles can give rise to theoretically interesting and practically important problems. They are dealt with, as a rule, by specialists known as aeroelasticians. Two

well-known books in this field are those written by Fung[33] and by Bisplinghoff, Ashley and Halfman[34]. Only one paper falling into this category will be presented at this conference, namely the one prepared by Y. C. Fung and M. Olson.

Buckling in the testing machine

Of the many possible time-dependent loading conditions not yet mentioned, one, buckling under the conditions prevailing in the ordinary testing machine, presents special interest. In industry, most compressed structural elements are designed on the basis of Euler's theory of buckling, or with the aid of one of the modifications of Euler's theory to account for inelastic behavior. The practical suitability of these theories is judged, as a rule, on the basis of a comparison with buckling loads obtained in the conventional mechanical or hydraulic testing machine. In 1949 the author[35] drew attention to the fact that the behavior of the dynamic system consisting of testing machine and test column does not necessarily agree with that of a compressive element in an airplane hitting the ground or in a bridge subjected to dead and live loads; nor do the initial and boundary conditions assumed in Euler's theory agree with those prevailing in the testing machine.

The process of buckling in the testing machine was therefore investigated with the aid of the dynamic equations of motion[35],[36],[37]. In the analysis, the testing machine was idealized to be perfectly rigid and its loading head was assumed to be descending at a constant velocity c. The initial deviations of the center line of the column from the straight

line of action of the compressive load were represented by a half sine wave of amplitude $e\rho$, where ρ is the radius of gyration of the cross section (Fig. 7). Under these conditions the amplitude $y_{middle} = F = \rho f$ varies as a function of the non-dimensional time $\xi = (1/\epsilon_E)(ct/L)$ as shown in Fig. 8. Here t is the time, L the length of the column and ϵ_E , defined as

$$\epsilon_E = \pi^2/(L/\rho)^2 \tag{2}$$

in is the Euler strain, that is the compressive strain (counted positive) that corresponds to Euler's buckling load with a perfectly straight and elastic column. The figure corresponds to a loading that is much more rapid than can ever be achieved in the conventional testing machine. The similarity number of loading Ω is given by

$$\Omega = \pi^2 \epsilon_E^3 (E/\mu c^2) \tag{3}$$

where μ is the mass per unit volume of the material of the column, and thus $(E/\mu)^{1/2} = a$ is the speed of sound in the material of the column. For steel and for aluminum alloys this speed is about 200,000 in. per second.

To obtain a representative value for Ω in rapid loading in the conventional testing machine, one may assume that a perfectly straight and elastic column of 10 in. length and of a slenderness ratio $L/\rho = 100$ is brought up to its Euler load in 100 seconds of testing. In this case the Euler strain ϵ_E is about 10^{-3} . The total displacement of the loading head $\epsilon_E L = 10^{-2}$ in. divided by 100 seconds gives $c = 10^{-4}$ in. per second. With $\pi^2 \simeq 10$ and $E/\mu = 4 \times 10^{10}$ (in./sec)^2 ,

Eq. (4) yields $\Omega \simeq 4 \times 10^{10}$. Buckling of the same column in 10 seconds, a speed seldom if ever reached in conventional testing, would reduce Ω to 4×10^{8} . Hence the value of 2.25 of the figure represents a much higher slenderness ratio or a much higher speed of loading than those of conventional tests. To obtain a comparable value of Ω in an actual test, the loading machine would have to be sped up to such an extent as to cause buckling in one-thousandth of a second; then, under the conditions given, the value of Ω would be 4 .

As Fig. 8 shows, under such rapid loading the lateral displacements of the column lag behind those calculated from static considerations. As a consequence, the load supported by the column can exceed the Euler load considerably[38]. Figure 9, taken from a report by Erickson, Nardo, Patel and Hoff[39], plots the theoretical values of the ratio of the maximum load to the Euler load as a function of the similarity number of loading Ω and of the non-dimensional initial deviation amplitude e. The circles in the diagram indicate experimental results obtained in a specially designed and constructed rapid loading machine. Columns with three values of the initial deviations were tested, namely with $e = 10^{-1}$, 10^{-2} and 10^{-3} . These values were maintained with a tolerance of ±10 per cent. The agreement between theory and experiment as shown in Fig. 9 is good.

It should be mentioned that the analysis loses its validity when the time necessary to reach the maximum load becomes so short that it is comparable to the time required for a pressure wave to travel from one end of the column to the other. This is not the case in the examples discussed. In one-thousandth of a second the pressure wave travels 200 inches while the length of the column is only 10 inches.

The effect of the interaction between pressure propagation along the column and lateral displacements of the column was investigated by Sevin[40]. The buckling of thin-walled circular cylindrical shells in the testing machine was studied by Vol'mir and Agamirov[41],[42].

Buckling criteria

The practical meaning of stability

When an engineer has completed the analysis of the state of static or dynamic equilibrium of a structure or machine, he would like to know whether he can rely upon this equilibrium state in view of the unavoidable inaccuracies of the manufacturing processes and in the presence of disturbances of all sorts. The expert in applied mechanics should be able to furnish him with an answer. If an answer sufficient to the needs of the engineer can be given through the solution of a classical eigenvalue problem of the Eulerian type, the expert can indulge in mathematically elegant calculations and at the same time do a useful job for the engineer. If such an answer is insufficient for any reason, it is necessary that the problem be set up analytically in such a manner as to take into account external non-conservative loads, internal dissipative mechanisms, initial deviations from the exact shape, disturbances of a finite magnitude and whatever else it takes to obtain a satisfactory solution. After all, applied mechanics is a branch of the natural sciences, and not a chapter of pure mathematics.

A very simple example should illustrate the difference between practical and highly idealized conditions. A suitcase standing in an automobile is in stable equilibrium as long as the forward speed of the

car is constant on a smooth road, but it is likely to fall flat on the floor when the driver brakes rapidly. The practical problem of stability is not answered in this case by a solution based on the theory of small disturbances.

A broad criterion proposed

To cope with the exigencies of real life, the following definition of stability is therefore proposed:

> A structure is in a stable state if admissible finite disturbances of its initial state of static or dynamic equilibrium are followed by displacements whose magnitude remains within allowable bounds during the required lifetime of the structure.

The magnitude and the number of admissible disturbances must be determined from a statistical and probabilistic investigation of the environment in which the structure will be used and due consideration must be given to required safety standards and to economic factors. Calculations of this kind are now commonly carried out when the fatigue of airplanes is investigated in the presence of gust loads.

Allowable bounds of displacements are those which do not interfere with the proper functioning of the structure. The lifetime required for the structure is determined from considerations of an economic nature, or, stated in a form that is more popular today, from considerations of cost effectiveness. The pharaohs built the pyramids to last for eternity, but the buildings on Fifth Avenue in New York

are replaced every thirty years or so by more modern ones. Transport airplanes are generally required to last through 30,000 to 40,000 hours of flying.

In the analysis the probable deviations of the shape of the structure from the perfect shape, caused by inaccuracies of manufacture, must be taken into account. The same is true of inaccuracies of loading and variations in material properties.

The task set appears to be overwhelming and naturally it should be, and as a matter of fact it can be simplified substantially in almost every practical case. But simplifications should be undertaken only when they can be justified, and the analyst should not begin with the a priori notion that the stability of a practical structure can necessarily be defined in the Eulerian manner.

Examples

Some of these thoughts were expressed by the author in his Colston Paper of 1949[43]. To illustrate the behavior patterns of typical, not perfectly elastic systems, the simple model of Fig. 10 was devised. It consists of a rigid lever of length L pivoted at the bottom and loaded with a vertical force P at the top. The upper end of the bar is supported laterally by a spring which may have elastic, viscoelastic, elasto-plastic, or other properties. The stability of the system is investigated by imparting, at time $t = 0$, to the upper end of the bar a horizontal velocity v and by calculating the horizontal displacement u from the initial vertical position during the ensuing motion.

When the structure is subject to creep, a first approximation to its behavior can be had by attributing linearly viscoelastic properties to the spring of the model. The displacements u (normalized through division by the disturbance velocity v) are plotted against time in Fig. 11. Curve 1 corresponds to P = 0; the oscillations are represented by undamped sine curves in the absence of a compressive load.

In the figure, the compression increases as the number written beside the curve increases, and for all non-zero values of the compressive load the displacements tend to infinity as time tends to infinity. According to the classical criterion all viscoelastic columns are therefore unstable. However, for practical purposes a reinforced concrete column in a building on Fifth Avenue is still stable if it develops very large deflections only after a thousand years. Hence the dynamic analysis should be used to determine the maximum displacements that are likely to occur during the lifetime of the column; they have to be calculated from the probable values of disturbance and initial deviation from the perfect shape.

It should be noted that the creep deformations of metal structures are highly non-linear functions of the loads, and that in the presence of nonlinear creep the deformations increase without bounds at a finite value of the time, the so-called critical time. A survey of the theories of buckling in the presence of non-linear creep was given by the author[44] in 1958.

An even more complex behavior pattern is exhibited by structures in the presence of dry friction. In Fig. 12, curves 1 and 2 show the displacements of the upper end of the bar under the same subcritical

compressive load, but the friction is higher for curve 2 than for curve 1. In both cases the motion stops, after some oscillations, at some distance from the initial state of equilibrium. If this distance is too great for proper functioning, the structure must be considered unstable in the presence of the disturbance velocity v even though P is less than the Euler load P_E of the structure.

Curve 3 corresponds to the Euler load, and curves 4 and 5 to a load $P = 1.1P_E$. In spite of the critical and supercritical magnitudes of the loads the displacements tend to a limit if the friction is sufficiently high; the value of the fricition increases from curve 3 through curve 4 to curve 5. If these limiting displacements can be tolerated by the structure, stability in a broad sense exists at and above the Euler load. However, the displacements tend to infinity when the friction is too low and $P > P_E$; this is shown in curve 6.

Initially imperfect structures

In a very interesting discussion of the concept of stability for elasto-plastic structures, Drucker and Onat[45] stated that the classical linearized condition of neutral equilibrium was really not relevant to inelastic buckling. They also showed, through the analysis of models, that essentially the same kind of information could be obtained from a kinetic analysis of a perfect system as from a static analysis of an imperfect system, provided the loads were of a static or quasi-static nature. They indicated a preference for the imperfection approach because they found it generally simpler than the kinetic approach.

Whenever there is evidence that no new, unexpected information on stability can be gained from the kinetic method, evidently there is no need to go to the trouble of using it. Thus in his classical analysis Koiter[46],[47] showed that the large reduction in the buckling stress of circular cylindrical shells subjected to axial compression and of spherical shells subjected to external pressure is a consequence of the great sensitivity of these structures to small deviations from the perfect shape. This sensitivity is illustrated in Fig. 13 which is reproduced here from a paper by Madsen and the author[48]. A_{11}^{o} is the non-dimensional amplitude of the initial deviations w_0 of the median surface of the shell from the ideal circular cylindrical shape. These deviations were assumed to be defined by

$$w_0 = A_{11}^{o}\ t\ \cos(\pi x/L_x)\cos(\pi y/L_y) + A_{20}^{o}\ t\ \cos(2\pi x/L_x)$$

where t is the thickness of the wall of the shell, x and y are the coordinates in the axial and circumferential directions, and L_x and L_y are the wave lengths in the same directions. This sensitivity can be detected without the use of the kinetic approach.

The kinetic approach and the assumption of initial deviations were included simultaneously in an analysis carried out by the author[38] in his Wilbur Wright Memorial Lecture of 1953. The system studied was an initially slightly curved column whose material followed a cubic (non-linear) stress-strain law. If the testing machine is elastic, there are three equilibrium states corresponding to each displacement $\xi = ct/L\epsilon_E$ of the loading head in a limited range of the ξ values. This fact was already mentioned by von Kármán[49] in his doctoral dissertation in 1910.

For the particular value $\xi = 19$ within this range the lateral oscillations of the column following a disturbance can be characterized by the phase plane diagram of Fig. 14. The technique of drawing such diagrams is explained in detail in Stoker's book[50] on nonlinear vibrations. L is the length of the column, ρ the radius of gyration of the cross section, and L/ρ the slenderness ratio of the column. The amplitude of the lateral oscillations of the midpoint of the column is $a\rho$ and the initial distance of the midpoint from the line of action of the compressive force is $e\rho$. S characterizes the elasticity of the testing machine.

The curves are contour lines of constant total energy plotted over the plane of non-dimensional lateral displacement a (abscissa) and non-dimensional lateral velocity v/K^{**} (ordinate). The three approximately elliptic regions are hollows, and their bottom corresponds to stable equilibrium in the presence of small disturbances. The two sides of the figure are not symmetric with respect to the axis of ordinates because the column was initially slightly curved, with the initial nondimensional deviation amplitude $e = 0.01$. The state of the system at $\xi = 19$, reached through a quasi-static loading process, is represented by the lowest point in the central hollow, a little to the right of the origin of coordinates. If the middle of the column is now pushed to the left or to the right a distance of 2.5ρ , just beyond the saddle point, and then let loose, oscillations around the stable equilibrium states represented by the bottom of the hollows to the left or the right, respectively, follow. Oscillations past all the three stable equilibrium positions are also possible if the energy of the disturbance is sufficiently large.

The ordinary quasi-static loading analysis discloses only the existence of the stable states of equilibrium of the central and right-hand hollows, and the unstable state of equilibrium corresponding to the saddle point between these two hollows. Obviously the left-hand portion of the phase plane cannot be reached from the initial equilibrium state without a disturbance.

It is worthwhile to call attention now to a possible source of error in the use of the vibration method. It follows from the theory of small oscillations that the frequency of the natural vibrations of a system decreases and tends to zero as the loads acting on the system increase and approach their first critical value. The conclusion should not be drawn from this fact that the value of the buckling load can necessarily be detected experimentally by a corresponding method of exciting vibrations. When the two pivoted ends of the column are held a fixed distance apart, as in a rigid testing machine (see Fig. 7) with the loading head fixed, the compression varies noticeably with the displacement of the column. The solution of the nonlinear differential equation governing the vibrations is an elliptic function which degenerates into a trigonometric function as the amplitude of the vibrations approaches zero. For finite amplitudes the frequency f does not approach zero as P approaches P_E.

This can be seen from Fig. 15 in which β is the amplitude a of vibrations divided by the radius of gyration ρ of the section, and the relative frequency is the frequency b divided by the frequency b_o in the absence of compression, with

$$b_o = (\pi\rho/2L^2)(E/\mu)^{1/2} \tag{4}$$

and μ the density of the material of the column. Figure 14 is taken from the doctoral dissertation of Burgreen[51], a former student of the author. Burgreen's experiments confirmed his theoretical conclusions.

A phenomenon in which initial deviations from the exact shape play a paramount role is creep buckling. The initial deviations are magnified by creep as time passes, and with metals, whose strain rate-stress law is highly nonlinear, the solution of the nonlinear differential equations indicates that the lateral displacements increase without bounds as a critical value of the time is approached. Of course, this critical value decreases as the compressive load P acting on the column increases, and the critical time is zero when $P = P_E$.

The fundamental phenomenon just described gets lost if the structure is idealized to such an extent that it is considered perfect. Probably this is the reason why a very elegant analysis by Rabotnov and Shesterikov[52], in which the method of small perturbations was used, yielded results that are difficult to interpret physically, as was pointed out by the author[44],[53].

Conference papers on buckling criteria

It appears that the definition of buckling and buckling criteria will be discussed at this Conference by J. J. Stoker and by B. Budiansky, and possibly also by J. P. La Salle and by V. V. Bolotin.

The tangent modulus load

Statement of the problem

The dynamic analysis can be used to re-investigate the controversy about the tangent modulus load. In 1946 Shanley[54] published a paper under the title The Column Paradox and in 1947 he followed it up with a second publication entitled Inelastic Column Theory[55]. In these publications he described consecutive stages of the buckling process of a column in the testing machine during which the decrease in the compressive stress on the convex side of the column caused by bending is balanced or exceeded by the increase in the average compressive stress caused by the increasing displacements of the loading head of the testing machine. The consequence of such a process is that the compressive stress increases monotonically throughout the cross section, more rapidly on the concave side and more slowly on the convex side. In the absence of a reversal in the sign of the time derivative of the stress, that is, in the absence of unloading, the stress increments are proportional to the strain increments, and the factor of proportionality is the tangent modulus E_t defined as

$$E_t = d\sigma/d\epsilon \tag{5}$$

where σ is the stress and ϵ the strain in the column (Fig. 16). The value of E_t is determined from the conventional stress-strain diagram of a strain-hardening material. In the initial, perfectly elastic part of the stress-strain curve the tangent modulus E_t is equal to Young's modulus E.

As long as there is no unloading and the displacements are infinitesimal throughout the cross section, the column behaves as a perfectly elastic column with a modulus E_t. Hence Euler's buckling stress

$$\sigma_{cr\ E} = \frac{\pi^2 E}{(L/\rho)^2} \tag{6}$$

where L is the length of the column, ρ the radius of gyration of its cross section, and L/ρ the slenderness ratio of the column, is replaced by the tangent-modulus buckling stress

$$\sigma_{cr\ t} = \frac{\pi^2 E_t}{(L/\rho)^2} \tag{7}$$

In contrast, Considère, Engesser and von Kármán assumed that the average stress in the column remains constant during the buckling process. Under these conditions the bending of the column increases the stresses over part of the cross section and decreases them over the remainder of the cross section. When the stress increases, Eq. (1) is valid and the factor of proportionality is E_t. In the remainder of the section only the elastic part of the strain can be recovered and the stress-strain law for unloading is

$$E = (d\sigma/d\epsilon)_{\sigma=0} \tag{8}$$

where E is Young's modulus. Hence the buckling stress is

$$\sigma_{cr\ r} = \frac{\pi^2 E_r}{(L/\rho)^2} \tag{9}$$

where E_r is the reduced modulus whose value depends both on the stress-strain curve and the shape of the column cross section. For an idealized I section, consisting of two concentrated flanges of a total cross sectional area A and of a web of vanishingly small area, the reduced modulus is

$$E_r = 2EE_t/(E + E_t) \tag{10}$$

For a solid rectangular cross section the expression is

$$E_r = 4EE_t/(\sqrt{E} + \sqrt{E_t})^2 \tag{11}$$

These formulas and the buckling of columns above the elastic limit of the material were discussed in detail by the author in earlier publictions[38],[56].

In his papers, Shanley drew the conclusion that since the column is capable of buckling at the tangent modulus stress, which is always less than the reduced modulus stress, it will buckle at the tangent modulus stress and thus the classical critical stress of the reduced modulus formula is erroneous. He confirmed this condlusion with the aid of column tests whose results were closer to the curve representing the tangent modulus buckling stress than to that representing the reduced modulus buckling stress. It is to be noted that the difference between the values according to the two curves was always small.

In the discussion of Shanley's second paper von Kármán[57] attributed the differences in the critical values obtained through the two processes described to the absence of a unique stress-strain relationship when the stress is higher than the limit of elasticity of the material. He did not point out, however, that Shanley had posed a

problem completely different from the classical problem solved by von Kármán. In the classical problem the stability of a system, consisting of a strain-hardening column and a prescribed load, is investigated by Euler's method. Shanley's problem is the determination of the buckling load during a loading process which, incidentally, was not completely defined. It is not paradoxical that two different solutions are found for two different problems.

The solution of von Kármán's classical problem is unique and is still given by the reduced modulus formula (9). Its correctness can neither be corroborated nor disproved by the conventional laboratory test which is carried out under conditions related to but in many respects different from the conditions assumed in the classical theory.

In the conventional laboratory test there is only one characteristic quantity that can be and is observed by the practicing engineer: the maximal value of the load. This maximal value is designated in practice as the buckling load of the column.

The problem for the specialist in applied mechanics is then to investigate the loading process of the conventional column test. The difficulties of such an investigation are manifold. The testing machine has mass and elasticity and thus, together with the column, forms a very complex dynamic system. When the testing machine is very rigid compared to the column, and when the rate of descent of the loading head is rigidly controlled, the following idealization appears permissible: The moving head of the testing machine can be assumed to descend with a uniform velocity c .

A second difficulty arises from the fact that columns cannot be manufactured with perfect accuracy. Their centroidal line will always deviate more or less from the theoretical straight line. At the same time it is impossible in the laboratory to apply the compressive load perfectly centrally. In a theoretical investigation one can assume perfection, and this is done in the classical theory; but the loading process in the laboratory can be reproduced analytically only if the practical deviations from the ideal system are given due consideration. When the results have been obtained for the imperfect system, it makes good sense to inquire how the results would change if the imperfections were made to decrease in magnitude and approach zero.

In his investigation Shanley assumed that he was free to control independently the compressive load acting on the column and the bending of the column. In this manner he was able to obtain lateral displacements, which under idealized conditions mean buckling, without stress reversal. (By stress reversal the practicing engineer means a change in the sign of the time derivative of the stress.) This led to the buckling stress associated with the tangent modulus.

The purpose of the present analysis is essentially to find out whether in the conventional laboratory test buckling can occur without such a stress reversal. For this purpose the test of an imperfect column in an idealized testing machine will be studied with the aid of the dynamic equations of motion.

From the standpoint of judging Shanley's work a significant shortcoming of the study presented here is that it is based on equations derived earlier by the author[35] for the dynamic buckling of perfectly

elastic columns. The results can only show therefore whether stress reversal precedes elastic buckling. Work is now in progress to extend the investigations to strain-hardening materials.

Differential equation and boundary and initial conditions

The integro-differential equation governing the lateral displacements of a perfectly elastic column tested in an idealized testing machine was derived in an earlier publication[35]. In the derivation it was assumed that the time required for a pressure wave to travel from one end of the column to the other is short compared to the time necessary for the lateral displacements to develop at buckling. Under such conditions inertia effects in the axial direction can be disregarded.

The initial deviations of the center line of the column (see Fig. 7) from the straight line along which the axial compressive load P is acting were assumed to be given by

$$y = y_o = \rho e \sin(\pi x/L) \qquad \text{when} \qquad t = 0 \tag{12}$$

where y_o is the initial deviation, e is the amplitude of the non-dimensionalized lateral deviation, ρ is the radius of gyration of the cross section of the column, x is the axial coordinate, measured from an endpoint of the column, and L is the length of the column.

The second initial condition was stated as

$$\partial y/\partial t = 0 \qquad \text{when} \qquad t = 0 \tag{13}$$

which means that the lateral velocity of points along the axis of the column is zero at the beginning of the experiment.

The boundary conditions at the two ends of the column were given by

$$y = \partial^2 y/\partial x^2 = 0 \qquad \text{when} \qquad x = 0,L \tag{14}$$

Under these conditions the assumption

$$y = \rho f \sin(\pi x/L) \tag{15}$$

where y is the total distance of the centroid of a cross section of the column from the straight line along which the load P is acting, and f is the amplitude of the non-dimensional distances at time t, satisfies the boundary conditions and makes it possible to carry out the integration. Thus the partial integro-differential equation is reduced to the following ordinary differential equation:

$$f'' + \Omega[(1-\xi)f - e + (1/4)f^3 - (1/4)e^2 f] = 0 \tag{16}$$

Here ξ is the non-dimensionalized time defined as

$$\xi = (1/\epsilon_E)(ct/L) \tag{17}$$

c is the uniform velocity of the downward motion of the loading head, t is the time elapsed from the beginning of the test, and the double prime $''$ indicates two successive differentiations with respect to ξ. The symbol ϵ_E is the Euler strain, that is the strain corresponding to Euler's critical stress:

$$\epsilon_E = \pi^2(\rho/L)^2 \tag{18}$$

The parameter Ω is defined as

$$\Omega = \pi^2 \epsilon_E^3 (E/\mu c^2) \tag{19}$$

with E Young's modulus and μ the mass of a unit volume of the material of the column.

The initial conditions can be written under these circumstances as

$$f = e \qquad f' = 0 \qquad \text{when} \qquad t = 0 = \xi \tag{20}$$

Another quantity of considerable interest is the value of the axial force P acting along the column at time t. It is given by the equation

$$P/P_E = \xi - (1/4)(f^2 - e^2) \tag{21}$$

Here P_E is the Euler load:

$$P_E = EA\epsilon_E = \pi^2 EI/L^2 \tag{22}$$

where A is the cross-sectional area and I the moment of inertia of the cross section of the column.

As it was assumed in the derivation of this equation that the curvature of the column could be represented by $\partial^2 y/\partial x^2$, the equation is valid only for moderately large lateral displacements; they should not exceed, say, one-twentieth of the length of the column. This means that for a column of a slenderness ratio $L/\rho = 100$ the non-dimensional amplitude f should be restricted to

$$|f| \leq 5 \tag{23}$$

Earlier solutions of the equations

The initial value problem represented by differential equation (16) and the initial conditions (20) was solved in the paper mentioned[35] for arbitrary values of Ω, but the numerical evaluation was carried

out only for relatively small values of Ω, that is for loading speeds much greater than those occurring in the conventional column test in the laboratory. It was found[35],[37] that with such high loading speeds the maximum load reached in the column can significantly exceed the Euler load.

As the loading speed is decreased, the non-dimensional displacement f becomes a monotonically increasing function of ξ upon which increasingly rapid oscillations are superimposed. For small displacements permitting a linearization of the equations, these oscillations are represented by Bessel functions of order 1/3 and -1/3 and of argument $(2/3)\Omega^{1/2}(1-\xi)^{3/2}$. Between $\xi = 1$ and $\xi = 0$ the value of the argument varies between zero and 666 if $\Omega = 10^6$ which still corresponds to a rather rapid loading process. In a slow loading process characterized by $\Omega = 10^{10}$, the range of the argument is about 666,666. In the former case there are about 200, and in the latter about 20,000 small oscillations between the beginning of the test and the time when the Euler displacement is reached.

In accordance with a suggestion of George F. Carrier, the slow loading process was studied in the Wilbur Wright Memorial lecture[38] with the aid of the B.W.K. method. It was established that rapid oscillations of small amplitude about the static solution constitute a good approximate solution of the problem; the number of oscillations found by this method agreed well with the numbers derived from the linearized solution valid at the beginning of the loading process.

Formulation for the new solution

Since the purpose of the present investigation is the determination of the times when stress reversal occurs and when the maximum load is reached in a slow loading process, another asymptotic solution appears to be advantageous. In the derivation of this new solution it is convenient to introduce the reciprocal of Ω :

$$\omega = 1/\Omega = (1/\pi^2 \epsilon_E^3)(\mu c^2/E) = (1/\pi^8)(c/a)^2(L/\rho)^6 \tag{24}$$

where a is the velocity of the propagation of sound in the material of the column. It is given by

$$a = (E/\mu)^{1/2} \tag{25}$$

Equations (16) and (20) defining the initial value problem can then be rewritten in the form

$$\left.\begin{aligned} &\omega f'' + (1-\xi)f - e + (1/4)f^3 - (1/4)e^2 f = 0 \\ &f = e \qquad df/d\xi = f' = 0 \qquad \text{when} \quad \xi = 0 \end{aligned}\right\} \tag{26}$$

This problem will have to be solved for small values of ω .

Maximum load and stress reversal

A necessary condition of a maximum of the load as given by Eq. (21) is

$$d(P/P_E)/d\xi = 0 \tag{27}$$

from which it follows that

$$ff' = 2 \tag{28}$$

This condition is equally valid, of course, for slow and for rapid loading, and for all cross-sectional shapes.

In the calculation of stress reversal it is convenient to consider only columns of idealized I section. They consist of two concentrated flanges, each of area $A/2$, held a distance $h = 2\rho$ apart by a web of vanishingly small cross-sectional area. It was shown in an earlier paper[58] that the behavior of such columns resembles that of the practical metal columns of engineering much more than does the behavior of the columns of solid rectangular section usually investigated in the literature. With the notation of the present paper the stress σ on the convex side of the column midway between the two end supports is

$$\sigma_{conv} = (P/A) - (M\rho/I) \tag{29}$$

With $I = A\rho^2$ and $M = -EI\left[\partial^2(y-y_o)/\partial x^2\right]$ this reduces to

$$\sigma_{conv} = (P/A) - (P_E/A)\ (f-e) \tag{30}$$

A necessary condition of stress reversal is

$$d\sigma/d\xi = 0 \tag{31}$$

which yields the equation

$$f'(2+f) = 2 \tag{32}$$

This equation is again valid for slow as well as rapid loading, but only for columns of ideal I section.

Static loading

When the loading takes place extremely slowly, ω can be set equal to zero. Equations (26) become

$$(1-\xi)f_o - e + (1/4)f_o^3 - (1/4)e^2 f_o = 0$$

$$f_o = e \qquad \text{when} \quad \xi = 0 \tag{33}$$

where the subscript o indicates static conditions. Differentiation with respect to ξ and solution for f_o' yield

$$f_o' = \frac{f_o}{1 - \xi + (3/4)f_o^2 - (1/4)e^2} \tag{34}$$

But Eq. (33) can be easily solved for ξ in terms of f_o :

$$1 - \xi = (e/f_o) - (1/4)f_o^2 + (1/4)e^2 \tag{35}$$

Substitution in Eq. (34) yields

$$f_o' = \frac{2f_o^2}{2e + f_o^3} \tag{36}$$

Substitutions in Eq. (21) lead to

$$P/P_E = \frac{f_o - e}{f_o} \tag{37}$$

The derivative of this expression is

$$P'/P_E = \frac{2e}{2e + f_o^3} \tag{38}$$

The condition of a maximum of the load, given in Eq. (28), becomes upon substitution from Eq. (36)

$$\frac{2f_o^3}{2e + f_o^3} = 2 \tag{39}$$

With a non-vanishing e this is satisfied only as f_o tends to infinity. On the other hand, the condition of stress reversal expressed in Eq. (32) has a solution in the finite range of f_o . Equation (32) becomes upon substitution and solution

$$f_{o\ rev} = e^{1/2} \tag{40}$$

where the subscript o rev indicates stress reversal during static loading. Substitution in Eq. (35) yields

$$1 - \xi_{rev} = e^{1/2} - (1/4)e + (1/4)e^2 \tag{41}$$

When e is small, this becomes

$$1 - \xi_{rev} \cong e^{1/2} \tag{42}$$

Thus stress reversal takes place close to, but below, $\xi = 1$ when e is small, and the smaller e is, the closer the value of ξ_{rev} is to 1 .

Since e is a small number for practical columns, with values ranging perhaps from 10^{-3} to $1/4$, stress reversal in very slow loading occurs at values of f_o less than unity, while the maximum load, that is the buckling load of the practical engineer, is reached only as f_o approaches infinity, as long as e is a non-zero quantity however small. Hence the stress reverses before the buckling load is reached.

In the limit, as e tends to zero, Eq. (35) reduces to

$$\xi - 1 = (f_o/2)^2 \qquad f_o \neq 0 \tag{43}$$

This equation has no real solution when $\xi < 1$. The only possible

solutions are therefore

$$f_o = 0 \qquad \text{when} \quad \xi = 1$$
$$f_o = 2(\xi-1)^{1/2} \qquad \text{when} \quad \xi \geq 1 \tag{44}$$

In the singular problem corresponding to $e = 0$ the column remains perfectly straight until $\xi = 1$ is reached. Beyond $\xi = 1$ two solutions exist. One is obtained if both e and f_o are set equal to zero in Eq. (33). In the other, f_o increases parabolically with increasing values of $\xi - 1$. The load increases linearly from zero to P_E when ξ increases from zero to one. From then on it remains constant, as can be seen from Eq. (37), if lateral displacements develop. Similarly, Eq. (38) shows that P' is zero, that is, the load versus end displacement curve is horizontal wherever f_o does not vanish, that is, when $\xi \geq 0$. Since for the perfect column, $e = 0$, lateral displacements can begin only at $\xi = 1$, and if they do begin they develop under constant load P, evidently stress reversal occurs at the instant of buckling.

To sum it up, in the real case of small initial deviations e the maximum load, that is the buckling load, is reached only when the displacements f_o become very large. Stress reversal, however, occurs before $\xi = 1$, and the value of $\xi = \xi_{rev}$ at the instant of reversal approaches unity as e is decreased and is allowed to tend to zero. In the highly artificial case of the perfect column, $e = 0$, one obtains the result that the column remains straight and the stress remains uniform until the time when $\xi = 1$. At that instant displacements may begin to occur, but in th at case the stress is reversed in the convex flange (Fig. 17).

A few other expressions of interest are:

(1) At $\xi = 0$

$$f_o = e \qquad f'_o = \frac{e}{1 + (1/2)e^2} \tag{45}$$

When e is small, the last expression becomes

$$f'_o \simeq e \tag{46}$$

(2) At $\xi = 1$

Equation (23) reduces to

$$e^2 f_o + 4e - f_o^3 = 0 \tag{47}$$

Solution of the quadratic yields

$$e = (2/f_o)\left\{-1 + \left[1 + \left(f_o^4/4\right)\right]^{1/2}\right\} \tag{48}$$

When $f_o^4/4 \ll 1$, this becomes

$$e \simeq f_o^3/4 \tag{49}$$

Hence

$$f_o \simeq (4e)^{1/3} \tag{50}$$

At the same time

$$f'_o \simeq \frac{4^{2/3}}{3e^{1/3}} \tag{51}$$

$$P/P_E \simeq 1 - (1/4)e^{2/3} \qquad P'/P_E \simeq 1/3 \tag{52}$$

(3) For use in the next section it is necessary to calculate the second derivative of f_o twice with respect to ξ. The result is:

$$f_o'' = 4f_o^3 \frac{4e - f_o^3}{(2e+f_o^3)^3} \tag{53}$$

Slow loading

In the case of slow loading it is worthwhile to attempt a solution in the form

$$f = f_o + \omega f_1 + \omega^2 f_2 + \ldots \qquad \omega \ll 1 \tag{54}$$

Substitution in the differential equation (26) and consideration of the fact that f_o satisfies Eq. (33) lead to the equation

$$f_o'' + f_1\left[1 - \xi + (3/4)f_o^2 - (1/4)e^2\right] = 0 \tag{55}$$

Solution for f_1 gives

$$f_1 = \frac{-f_o''}{1 - \xi + (3/4)f_o^2 - (1/4)e^2} \tag{56}$$

Substitution of quantities determined earlier leads to

$$f_1 = -8f_o^4 \frac{4e - f_o^3}{(2e+f_o^3)^4} \tag{57}$$

Again, one can calculate the derivative with respect to ξ:

$$f_1' = -16f_o^5 \frac{32e^2 - 46ef_o^3 + 5f_o^6}{(2e+f_o^3)^6} \tag{58}$$

Similarly

$$f_1'' = 16f_o^6 \frac{-640e^3 + 2304e^2f_o^3 - 1140ef_o^6 + 70f_o^9}{(2e+f_o^3)^8} \tag{59}$$

If terms multiplied by ω^2 are next taken into account, one obtains the equation

$$f_1'' + (1-\xi)f_2 + (3/4)f_of_1^2 + (3/4)f_o^2f_2 - (1/4)e^2f_2 = 0 \tag{60}$$

Hence

$$-f_2 = \frac{f_1'' + (3/4)f_of_1^2}{1 - \xi + (3/4)f_o^2 - (1/4)e^2} \tag{61}$$

This can also be written as

$$-f_2 = 2f_o \frac{f_1'' + (3/4)f_of_1^2}{2e + f_o^3} \tag{62}$$

or as

$$f_2 = \frac{32f_o^7}{(2e+f_o^3)^9}\left(640e^3 - 2352e^2f_o^3 + 1164ef_o^6 - 73f_o^9\right) \tag{63}$$

The derivative of f_2 with respect to ξ is

$$f_2' = \frac{64f_o^8}{(f_o^3+2e)^{11}}\left(803f_o^{12} - 18{,}632ef_o^9 + 70{,}248e^2f_o^6 - 59{,}840e^3f_o^3 + 8960e^4\right) \tag{64}$$

In the calculation carried out in this section the successive terms of the asymptotic series have been obtained without any consideration of the initial conditions. It is necessary to check therefore what kind of initial conditions the solution implies.

For small values of e, the nondimensional initial velocity $f_o' \simeq e$. If it remained constant, it would cause a nondimensional lateral displacement equal to e by the time the Euler displacement is reached ($\xi = 1$). It is satisfactory therefore for our purpose to investigate a column with the initial conditions $f_o = e$ and $f_o' = e$ instead of $f_o = e$ and $f_o' = 0$. The higher order approximations f_1 and f_2 yield quantities proportional to and of the order of e for initial displacement and velocity. As these must be multiplied by $\omega, \omega^2, \ldots,$ their effect on the practical initial conditions can be disregarded.

The equations derived in this section were used to determine the history of loading for a number of combinations of the values of ω and e. Part of the results obtained on the Burroughs B5500 digital computer of the Computation Center of Stanford University are shown in Figs. 18 and 19. Only the portion of the history near $\xi = 1$ is presented because elsewhere the various curves are practically identical with one another and with the curves representing infinitely slow loading ($\omega = 0$).

When $\omega \leq 10^{-7}$ (where the equal sign corresponds to buckling in 1.58 sec if the slenderness ratio is 100), the dynamic displacements practically coincide with the static displacements, as can be seen from Fig. 18. For higher speeds of loading systematic differences arise; unfortunately the magnitude of the deviations from those characterizing

static conditions are so great that the validity of the asymptotic solution is in doubt. The dynamic curves of Fig. 18 must be considered only as an indication of the effects of higher loading speeds but the numerical values should not be relied upon.

In Fig. 19 the curve labeled $\omega = 10^{-7}$ coincides with the static curve. Its maximal value is reached asymptotically as ξ approaches infinity. With the higher speeds maxima occur in the neighborhood of $\xi = 1$. All the values shown in the two figures correspond to $e = 10^{-3}$.

The values of $f = f_{rev}$ and $\xi = \xi_{rev}$ corresponding to stress reversal can be calculated without difficulty for all the cases investigated. Some of the numerical results are collected in Table 1.

TABLE 1

Values of ξ_{rev} and $\xi_{P=P_{max}}$ for $e = 10^{-3}$
and for various loading speeds

ω	ξ_{rev}	$\xi_{P=P_{max}}$
1.5×10^{-6}	0.969	0.989
10^{-6}	0.969	0.995
10^{-7}	0.969	None
10^{-8}	0.969	None
10^{-9}	0.969	None
10^{-10}	0.969	None

The variation of ξ_{rev} with e for a fixed value of the loading speed, namely $\omega = 10^{-7}$, is given in Table 2. Values of $\xi_{P=P_{max}}$ are not listed as they could not be detected in the neighborhood c. $\xi = 1$. It must be assumed that the load curve rises monotonically and approaches $P/P_E = 1$ as ξ tends to infinity.

TABLE 2

Values of ξ_{rev} for $\omega = 10^{-7}$ and for various initial deviation amplitudes e

$e =$	10^{-1}	10^{-2}	10^{-3}	10^{-4}
$\xi_{rev} =$	0.7106	0.9025	[illegible] 9	0.9880
$\xi_{rev\ static} =$	0.7065	0.9025	0.9686	0.9900

It is concluded therefore that both in static loading and at the usual speeds of the conventional testing machine, stress reversal precedes the maximal load value when the column is perfectly elastic, except that stress reversal and maximum load coincide for quasi-static loading in the limit as the initial deviation from straightness approaches zero. The same conclusions were drawn earlier in the Wilbur Wright Lecture[38] on the basis of oscillographic recordings of tests carried out in conventional testing machines. Further corroboration of these conclusions is contained in a set of diagrams published for strainhardening columns by Wilder, Brooks and Mathauser[59]. These results were obtained for an idealized I-section with the aid of a digital computer.

It is worth noting that the fact that buckling, as defined by the practical engineer, in other words the attainment of the maximum load in the testing machine, cannot take place before stress reversal, can be derived directly from Eqs. (28) and (32) in the case of an elastic column. Indeed, both f and f' are always positive in quasi-static loading; this follows from the solution of Eqs. (33). Moreover, f increases monotonically with ξ under these conditions. Hence Eq. (32) will always be satisfied before Eq. (28) is satisfied. The same can be said for loading at the usual speeds of the conventional testing machine because the dynamic analysis has shown that the perturbation of the quasi-static state is negligibly small under these conditions.

Conclusions

Ziegler[5] concluded his paper in Advances in Applied Mechanics by stating that in difficult problems the stability analysis must be carried out with the aid of the kinetic criterion, and that initial deviations from the exact shape must also be taken into account. The writer of the present paper agrees with Ziegler. Whenever a simpler analysis will obviously yield the correct answer, the more difficult approach can be dispensed with, as was argued by Drucker and Onat[45]. But there will always arise questions that cannot be answered with assurance unless a complete analysis in the above sense is undertaken.

Fortunately this complete analysis can be undertaken now whenever necessary. With further developments in applied mathematics and in computer design and techniques many problems that even now appear formidable will be reduced in difficulty to acceptable levels.

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References

1. R. Fuchs and L. Hopf, Aerodynamik, Berlin, 1922

2. G. Herrmann and R. W. Bungay, On the Stability of Elastic Systems Subjected to Nonconservative Forces, Journal of Applied Mechanics, 31, 435 (1964)

3. C. B. Biezeno and R. Grammel, Engineering Dynamics, First English Edition, Blackie and Son, London (1955); First German Edition entitled Technische Dynamik, Julius Springer, Berlin (1939)

4. Hans Ziegler, Die Stabilitätskriterien der Elastomechanik, Ingenieur-Archiv, 20, 49 (1952)

5. Hans Ziegler, On the Concept of Elastic Stability, Advances in Applied Mechanics, Academic Press, New York (1956) p. 351

6. V. V. Bolotin, The Dynamic Stability of Elastic Systems, translated by V. I. Weingarten, L. B. Greszczuk, K. N. Trirogoff and K. D. Gallegos, Holden-Day, Inc., San Francisco, 1964 (original Russian publication Gostekhizdat, Moscow, 1956; revised German translation VEB Deutscher Verlag der Wissenschaften, Berlin, 1961)

7. N. M. Beliaev, Stability of Prismatic Rods Subjected to Variable Longitudinal Forces, in Engineering Construction and Structural Mechanics, Put', Leningrad, 1924, p. 149

8. N. M. Krylov and N. N. Bogoliubov, Calculations of the Vibrations of Frame Construction with Consideration of Normal Forces and with the Help of the Methods on Nonlinear Mechanics, in Investigation of Vibration of Structures, ONTI Kharkov/Kiev, 1935, p. 5

9. S. Lubkin, Stability of Columns under Periodically Varying Loads, Doctoral Dissertation, New York University, New York, 1939

10. S. Lubkin and J. J. Stoker, Stability of Columns and Strings under Periodically Varying Forces, Quarterly of Applied Mathematics, 1, 215 (1943)

11. I. Utida and K. Sezawa, Dynamical Stability of a Column under Periodic Longitudinal Forces, Reports of the Aeronautical Research Institute of Tokyo Imperial University, 15, 193 (1940)

12. E. Mettler, Biegeschwingungen eines Stabes unter pulsierender Axiallast, Mitteilungen des Forschungs instituts des Gutehoffnunghütte-Konzerns, 8, 1 (1940)

13. R. Einaudi, Sulle configurazioni di equilibrio instabile di una piastra sollecitata da sforzi tangenziali pulsante, Atti Accademia Gioenia Catania, 20, 1 (1936) and 5, 1 (1937)

14. Carel Koning and Joseph Taub, Stossartige Knickbeanspruchung schlanker Stäbe im elastischen Bereich bei beiderseits gelänkiger Lagerung, Luftfahrtforschung, 10, 17 (1933)

15. Joseph Taub, Stossartige Knickbeanspruchung schlanker Stäbe im elastischen Bereich, Luftfahrtforschung, 10, 65 (1933); English translation entitled Impact Buckling of Thin Bars in the Elastic Range for Any End Condition, Technical Memorandums, National Committee for Aeronautics, No 749 (1934)

16. N. J. Hoff and Victor Bruce, Dynamic Analysis of the Buckling of Laterally Loaded Flat Arches, Journal of Mathematics and Physics, 32, 276 (1954)

17. E. I. Grigolyuk, Nonlinear Vibrations and Stability of Shallow Shells, Izvestia Akademii Nauk SSSR, 3, 33 (1955), translated in Applied Mechanics Series, 131 (1960) of Institute of Engineering Research, University of California

18. John S. Humphreys and Sol R. Bodner, Dynamic Buckling of Shallow Shells under Impulsive Loading, Proceedings of the American Society of Civil Engineers, Journal of the Engineering Mechanics Division, 88, 17 (1962)

19. Bernard Budiansky and John W. Hutchinson, Dynamic Buckling of Imperfection-Sensitive Structures, to be published in Proceedings of the Eleventh International Congress of Applied Mechanics, Springer, Berlin

20. Anthony P. Coppa, On the Mechanism of Buckling of a Circular Cylindrical Shell under Longitudinal Impact, Technical Information Series R60SD494 of the Space Sciences Laboratory of the General Electric Company, Missile and Space Vehicle Department (1960)

21. H. Schwieger and Chr. Spuida, Ueber das Beulverhalten von Kreiszylinderschalen bei schlagartiger Axialbelastung, Deutsche Luft- und Raumfahrt, Forschungsbericht 65-23 (1965)

22. Emmett A. Witmer, Hans A. Balmer, John W. Leech, and Theodore H. H. Pian, Large Dynamic Deformations of Beams, Rings, Plates and Shells, AIAA Journal, 1, 1848 (1963)

23. J. N. Goodier and I. K. McIvor, The Elastic Cylindrical Shell under Nearly Uniform Radial Impulse, Journal of Applied Mechanics, 31, 259 (1964)

24. H. E. Lindberg, Buckling of a Very Thin Cylindrical Shell due to an Impulsive Pressure, Journal of Applied Mechanics, 31, 267 (1964)

25. G. R. Abrahamson and J. N. Goodier, Dynamic Plastic Flow Buckling of a Cylindrical Shell from Uniform Radial Impulse, Proceedings of the Fourth U.S. National Congress of Applied Mechanics, American Society of Mechanical Engineers, New York, 939 (1962)

26. A. Pflüger, Stabilitätsprobleme der Elastostatik, Julius Springer, Berlin, Germany, 1950

27. Hans Ziegler, Die Stabilitätskriterien der Elastomechanik, Ingenieur-Archiv, 20, 49 (1952)

28. M. Beck, Die Knicklast des einseitig eingespannten, tangential gedrückten Stabes, Zeitschrift für Angewandte Mathematik und Physik, 3, 225 (1952)

29. Hans Ziegler, Zum Begriff des Konservativen Systems, Elemente der Mathematik, 7, 121 (1952)

30. V. V. Bolotin, Nonconservative Problems of the Theory of Elastic Stability, Moscow, 1961; English translation published by Pergamon Press, New York, 1963

31. E. L. Nikolai, On the Stability of the Rectilinear Form of Equilibrium of a Bar in Compression and Torsion, Izv. Leningr. politekhn. in-ta 31 (1928)

32. E. L. Nikolai, On the Problem of the Stability of a Bar in Torsion, Vestn. prikl. matem. mekh. 1 (1929)

33. Y. C. Fund, An Introduction to the Theory of Aeroelasticity, GALCIT Aeronautical Series, John Wiley and Sons, New York 1955

34. Raymond L. Bisplinghoff, Holt Ashley, and Robert L. Halfman, Aeroelasticity, Addison-Wesley, Cambridge, Mass., 1955

35. Nicholas J. Hoff, The Process of the Buckling of Elastic Columns, PIBAL Report No. 163, Polytechnic Institute of Brooklyn (1949). Also printed in Journal of Applied Mechanics, 18, 68 (1951)

36. N. J. Hoff, S. V. Nardo and Burton Erickson, An Experimental Investigation of the Process of Buckling of Columns, Proceedings of the Society for Experimental Stress Analysis, 9, 201 (1951)

37. N. J. Hoff, S. V. Nardo and Burton Eickson, The Maximum Load Supported by an Elastic Column in a Rapid Compression Test, Proceedings of the First U.S. National Congress of Applied Mechanics, the American Society of Mechanical Engineers, New York, 419 (1952)

38. N. J. Hoff, Buckling and Stability, The Forty-First Wilbur Wright Memorial Lecture, Journal of the Royal Aeronautical Society, 58, 3 (1954)

39. Burton Erickson, S. V. Nardo, Sharad A. Patel and N. J. Hoff, An Experimental Investigation of the Maximum Loads Supported by Elastic Columns in Rapid Compression Tests, Polytechnic Institute of Brooklyn Department of Aeronautical Engineering PIBAL Report No. 296 (1955)

40. Eugene Sevin, On the Elastic Bending of Columns Due to Dynamic Axial Forces Including Effects of Axial Inertia, Journal of Applied Mechanics, 27, p. 125 (1960)

41. A. S. Vol'mir, Ob ustoychivosti cilindricheskikh obolochek pri dinamicheskom nagruzhenii, Dokladi Akademii Nauk, SSSR, 123, 806 (1958)

42. V. L. Agamirov and A. S. Vol'mir, Povedenie cilindricheskikh obolochek pri dinamicheskom nagruzhenii vsestovonnego davleniya pri ocevogo szhatiya, Izvestiya Akademii Nauk, SSSR, 78 (1959); also Behavior of Cylindrical Shells under Dynamic Loading by Hydrostatic Pressure or by Axial Compression, ARS Journal, 31, 98 (1961)

43. N. J. Hoff, Dynamic Criteria of Buckling, Proceedings of a Symposium of the Colston Research Society on Engineering Structures, Butterworths Scientific Publications, London, and Academic Press, New York, 1949, p. 121

44. Nicholas J. Hoff, A Survey of the Theories of Creep Buckling, Proceedings of the Third U.S. National Congress of Applied Mechanics, The American Society of Mechanical Engineers, New York, 1958, p. 29

45. D. C. Drucker and E. T. Onat, On the Concept of Stability of Inelastic Systems, Journal of the Aeronautical Sciences, 21, 543 (1954)

46. W. T. Koiter, Over de Stabiliteit van het elastisch Evenwicht, H. J. Paris, Amsterdam, 1945

47. W. T. Koiter, Elastic Stability and Postbuckling Behavior, Proceedings, Symposium on Non-Linear Problems, edited by R. E. Langer, University of Wisconsin Press, 1963, p. 257

48. Wayne A. Madsen and Nicholas J. Hoff, The Snap-Through and Postbuckling Equilibrium Behavior of Circular Cylindrical Shells under Axial Load, Stanford University Department of Aeronautics and Astronautics Report SUDAER No. 227, April 1965

49. Theodore von Kármán, Untersuchungen über Knickfestigkeit, Mitteilungen über Forschungsarbeiten auf dem Gebiete des Ingenieurwesens, Verein Deutscher Ingenieure, Berlin, Heft 81, 1910

50. J. J. Stoker, Nonlinear Vibrations in Mechanical and Electrical Systems, Interscience Publishers, New York, 1950

51. David Burgreen, Free Vibrations of a Pin-Ended Column with Constant Distance between Pin Ends, Journal of Applied Mechanics, 18, 135 (1951)

52. G. N. Rabotnov and S. A. Shesterikov, Creep Stability of Columns and Plates, Journal of the Mechanics and Physics of Solids, 6, 27 (1957)

53. N. J. Hoff, Reversed Creep: A Remark to the Creep Buckling Theory of Rabotnov and Shesterikov, Journal of the Mechanics and Physics of Solids, 12, 113 (1964)

54. F. R. Shanley, The Column Paradox, Journal of the Aeronautical Sciences, 13, 678 (1946)

55. F. R. Shanley, Inelastic Column Theory, Journal of the Aeronautical Sciences, 14, 261 (1947)

56. N. J. Hoff, The Analysis of Structures, John Wiley and Sons, New York, 1956, p. 318

57. Th. von Kármán, Discussion of Inelastic Column Theory by F. R. Shanley, Journal of the Aeronautical Sciences, 14, 267 (1947)

58. N. J. Hoff, The Idealized Column, Festschrift Richard Grammel, Ingenieur-Archiv, 28, 89 (1959)

59. Thomas W. Wilder, III, William A. Brooks, Jr., and Eldon E. Mathauser, The Effect of Initial Curvature on the Strength of an Inelastic Column, National Advisory Committee for Aeronautics Technical Note 2872, Washington, 1953

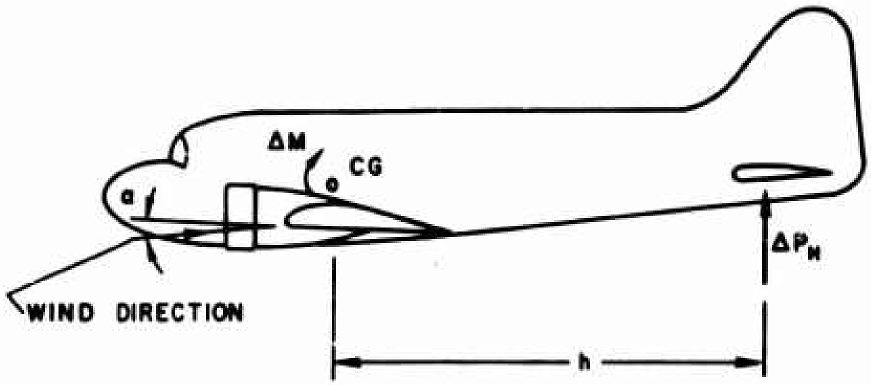

Fig. 1. Change in Moment Equilibrium of Airplane in Consequence of Change in Angle of Attack

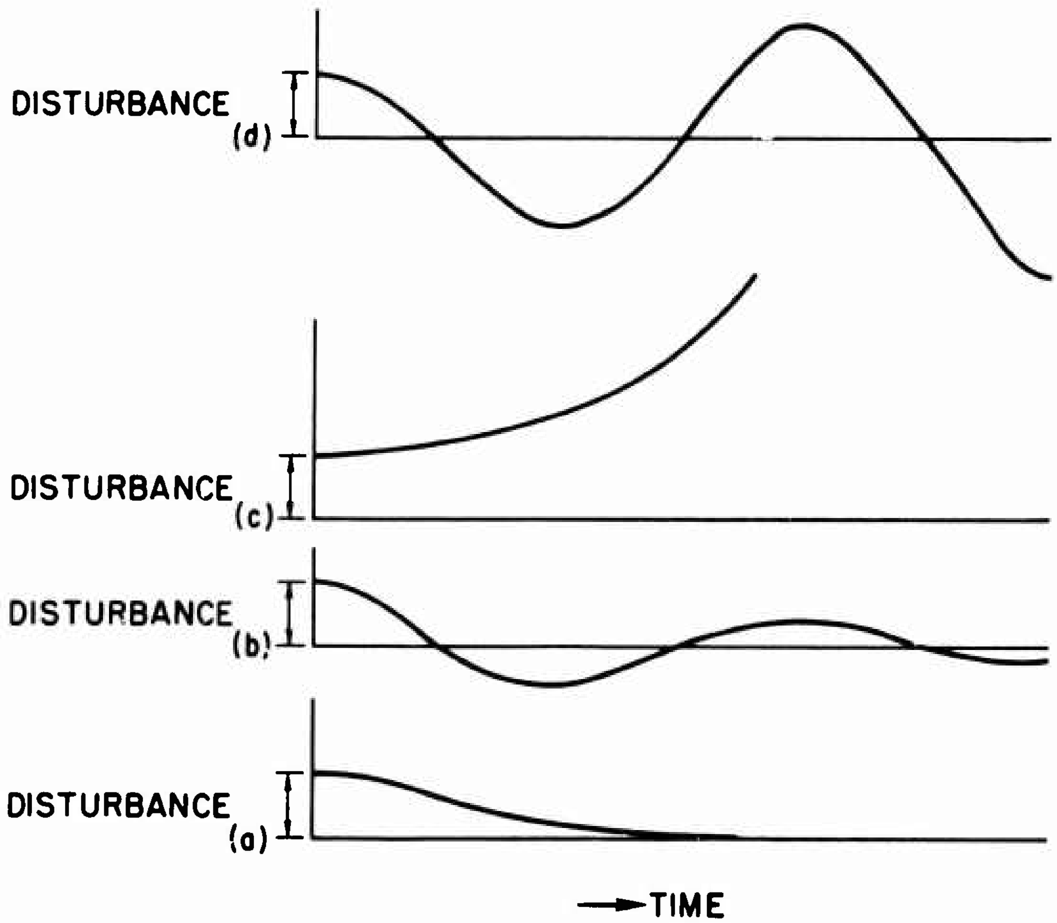

Fig. 2. Response of Airplane to Disturbance

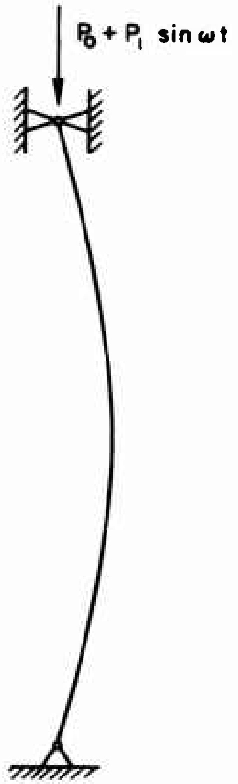

Fig. 3. Parametric Resonance

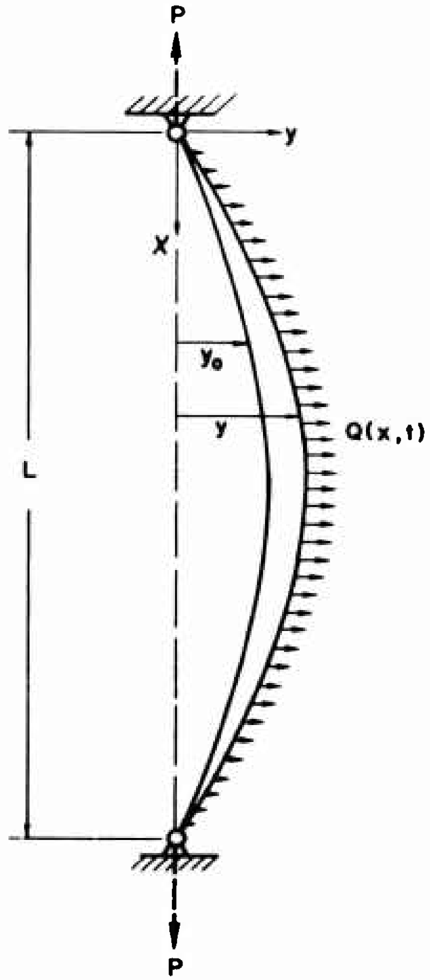

Fig. 4. Laterally Loaded Flat Arch (From Journal of Mathematics and Physics)

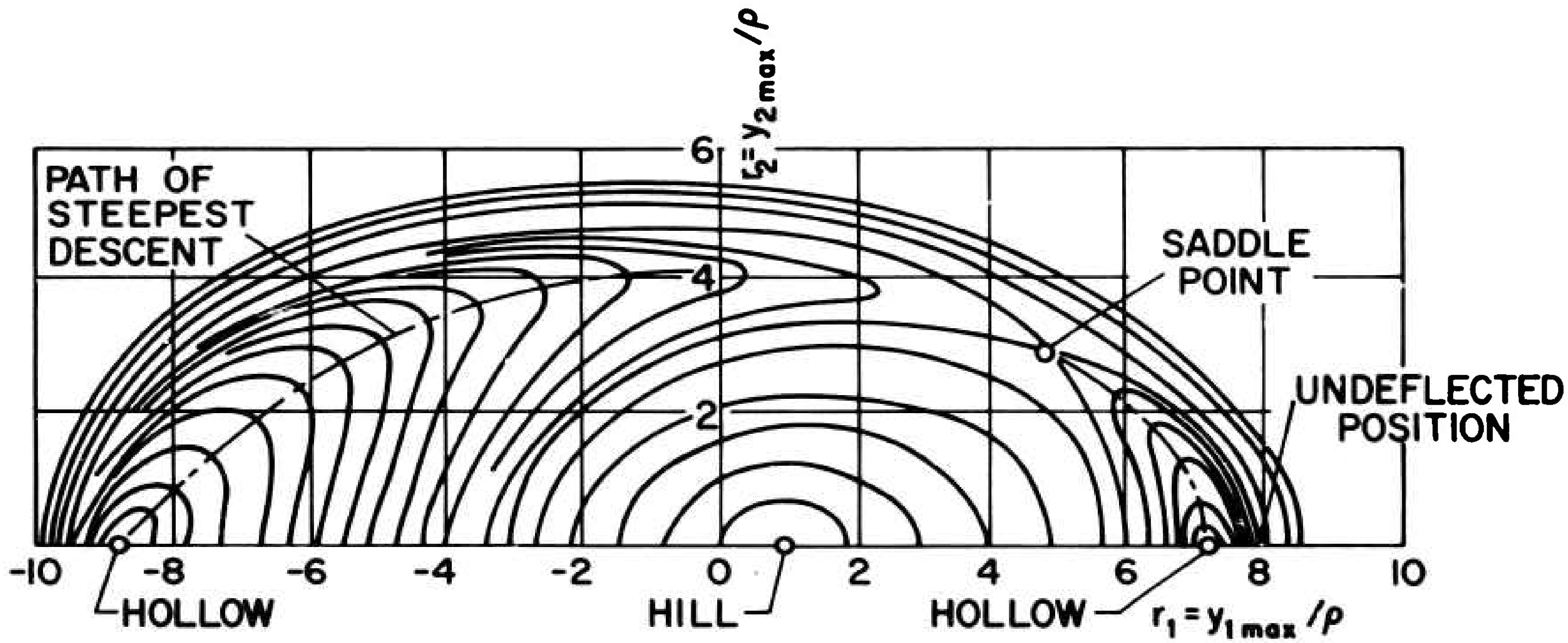

Fig. 5. Snap-Through of Laterally Loaded Flat Arch (From Journal of Mathematics and Physics

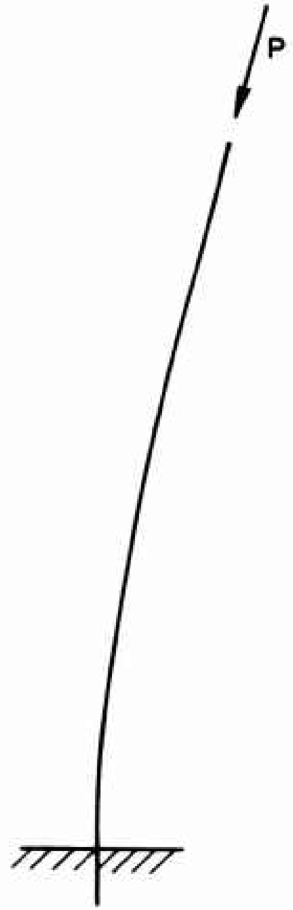

Fig. 6. Column under Tangential End Load

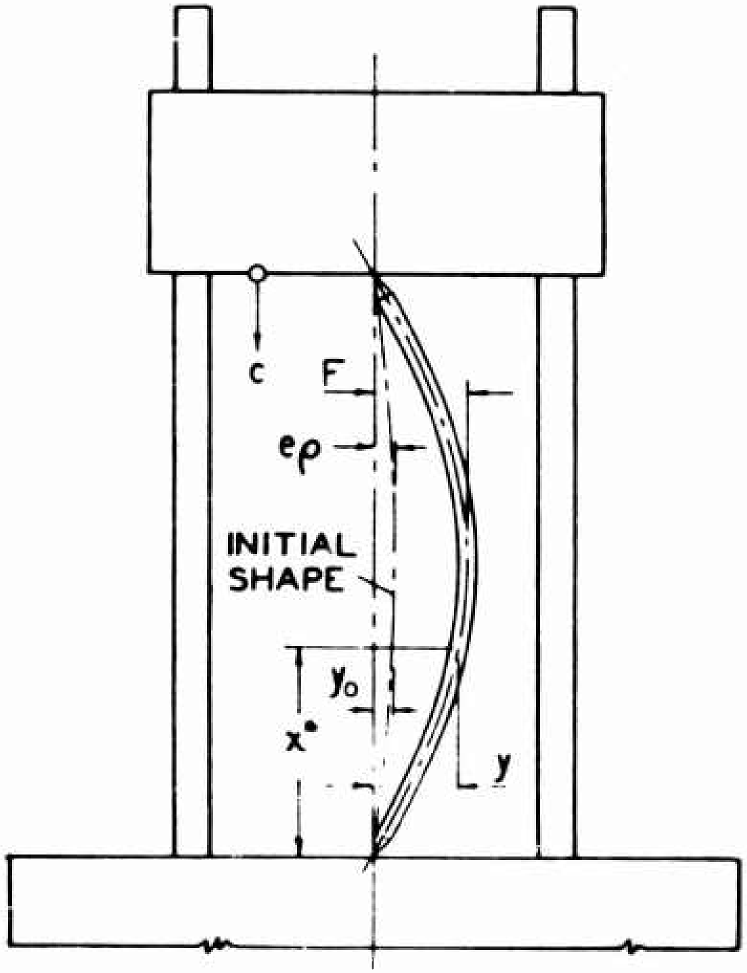

Fig. 7. Column in Conventional Testing Machine (From Journal of the Royal Aeronautical Society)

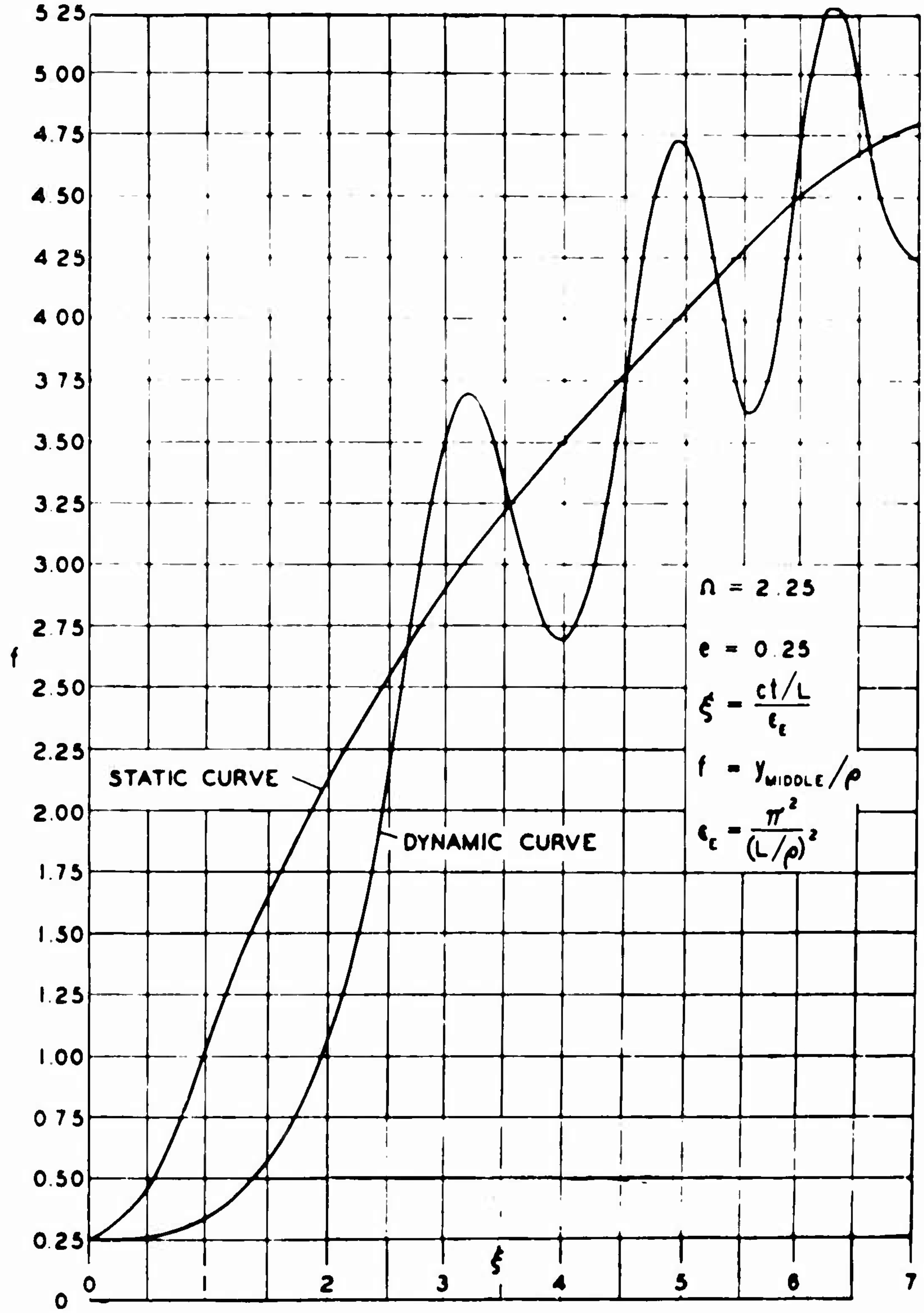

Fig. 8. Nondimensional Lateral Displacement Amplitude f as Function of Nondimensional Time in Very Rapid Loading (From Journal of the Royal Aeronautical Society)

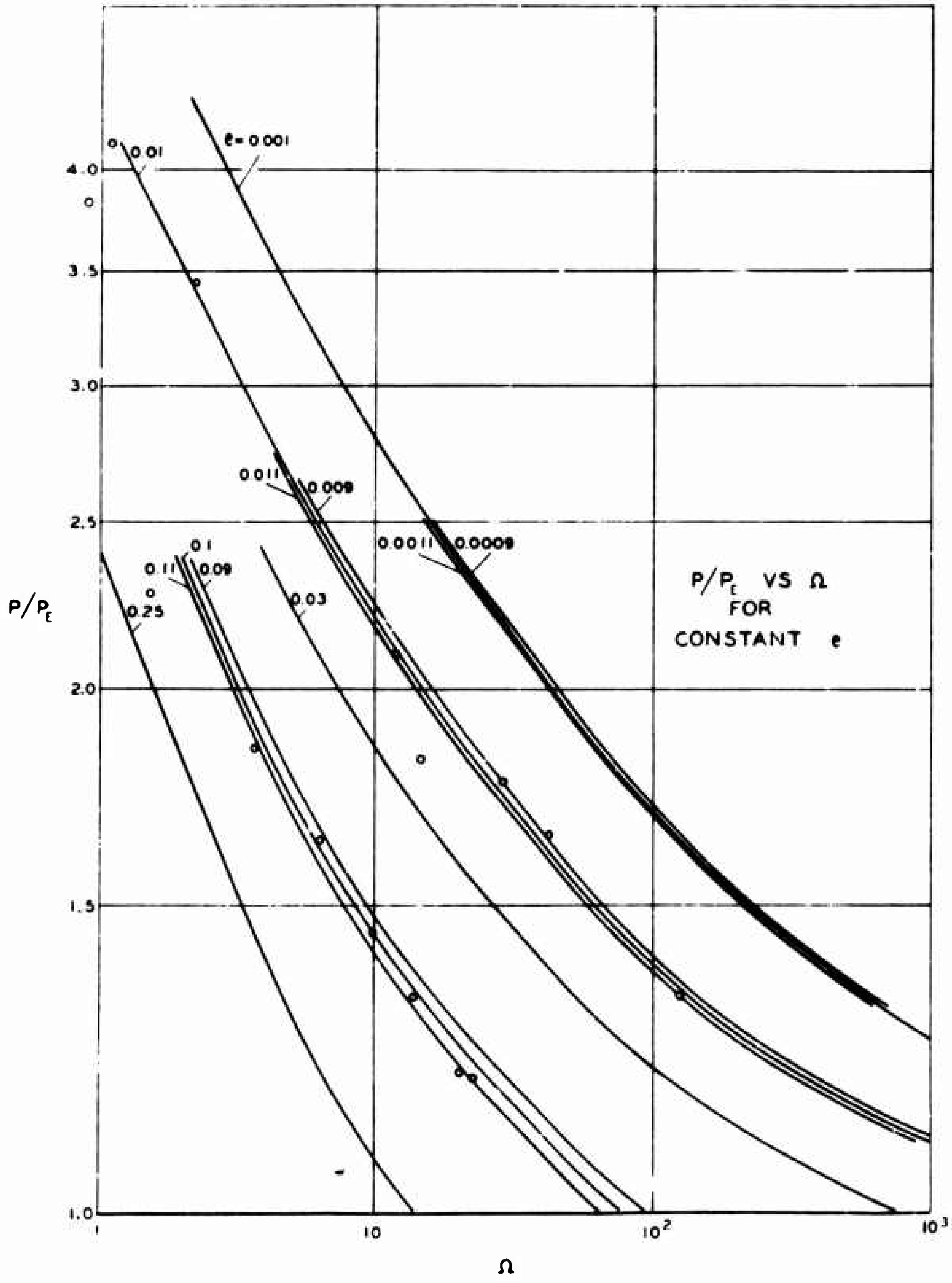

Fig. 9. End Load Ratio as Function of Nondimensional Time in Very Rapid Loading (From PIBAL Report 296 of the Polytechnic Institute of Brooklyn Department of Aeronautical Engineering)

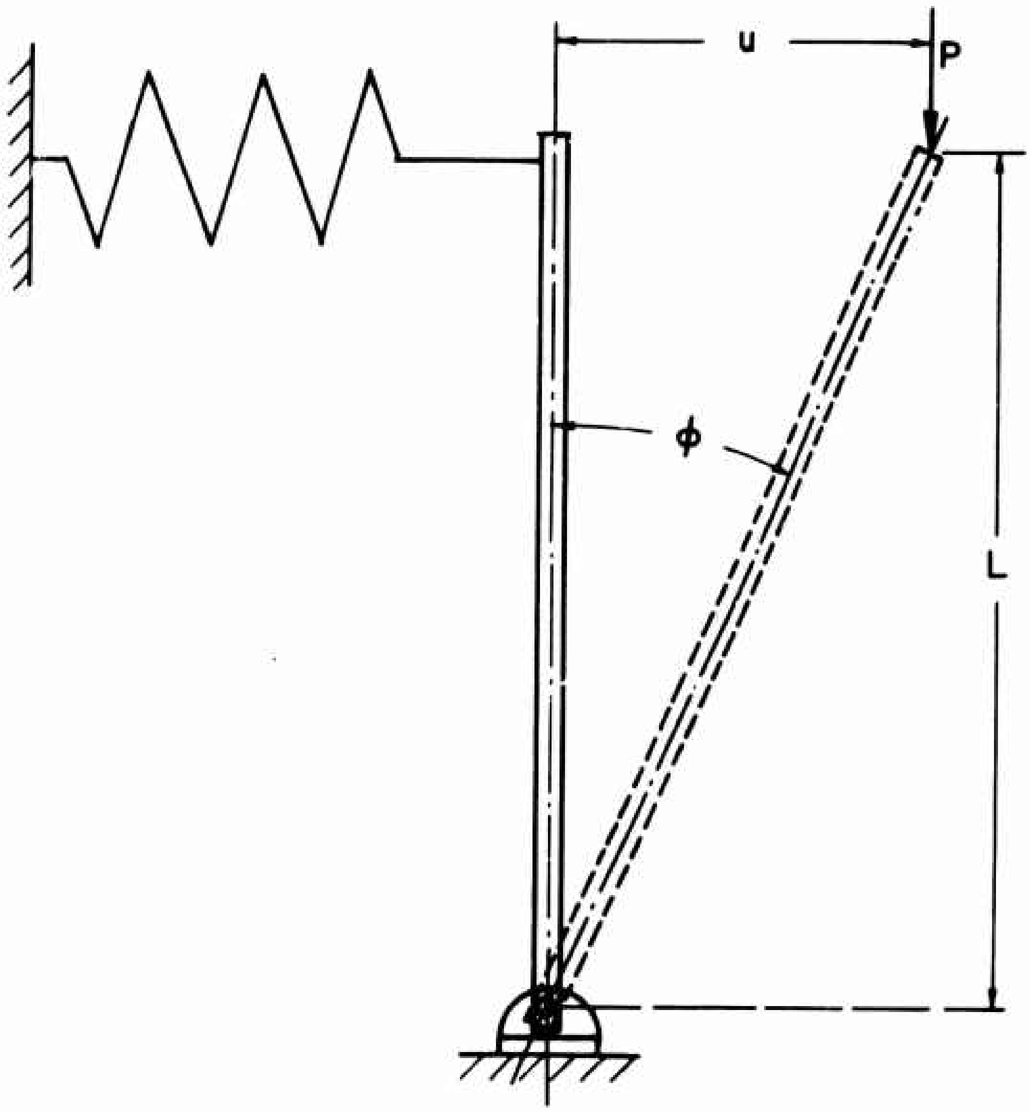

Fig. 10. Buckling Model (From Symposium of the Colston Research Society on Engineering Structures)

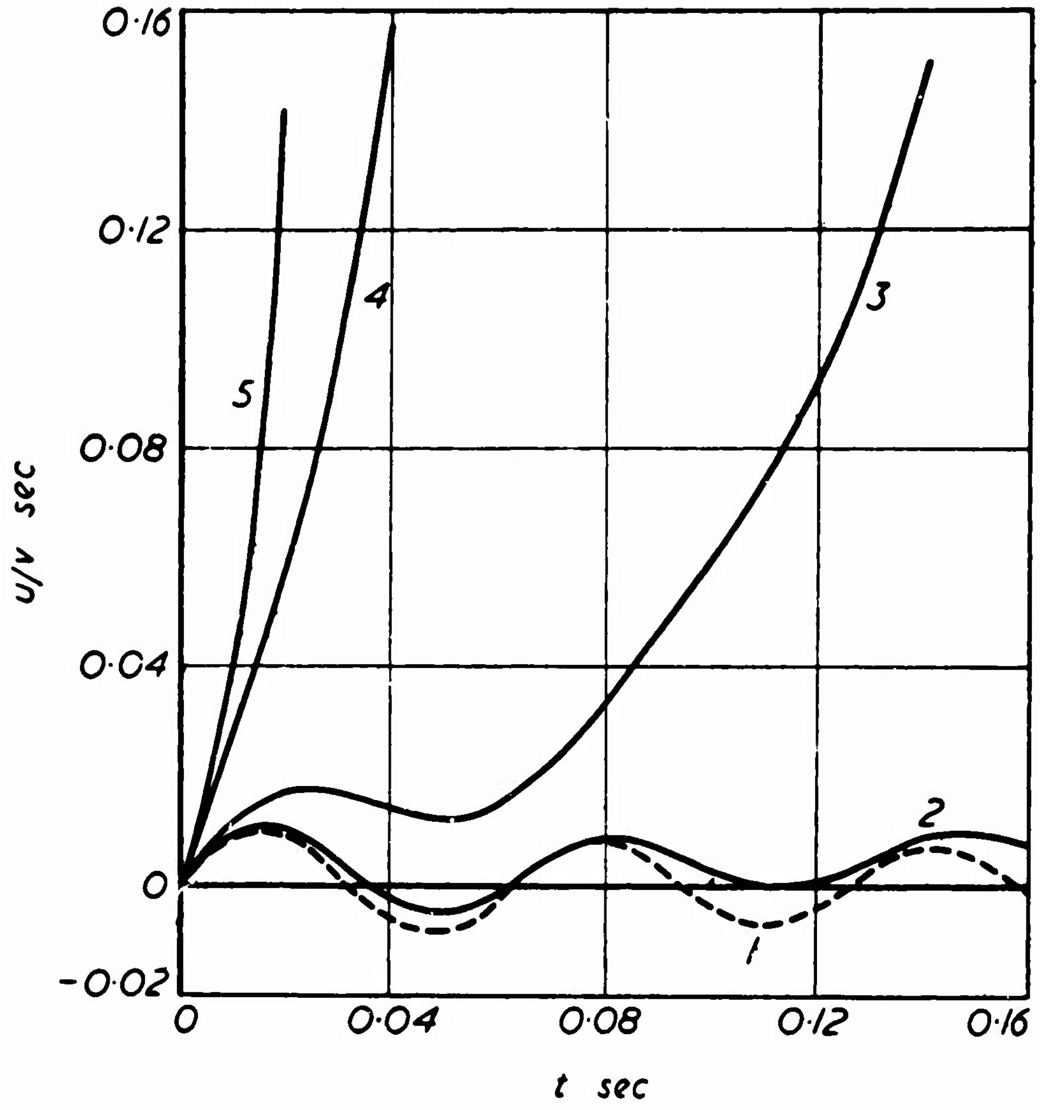

Fig. 11. Displacements of Linearly Viscoelastic Model (From Symposium of the Colston Research Society on Engineering Structures)

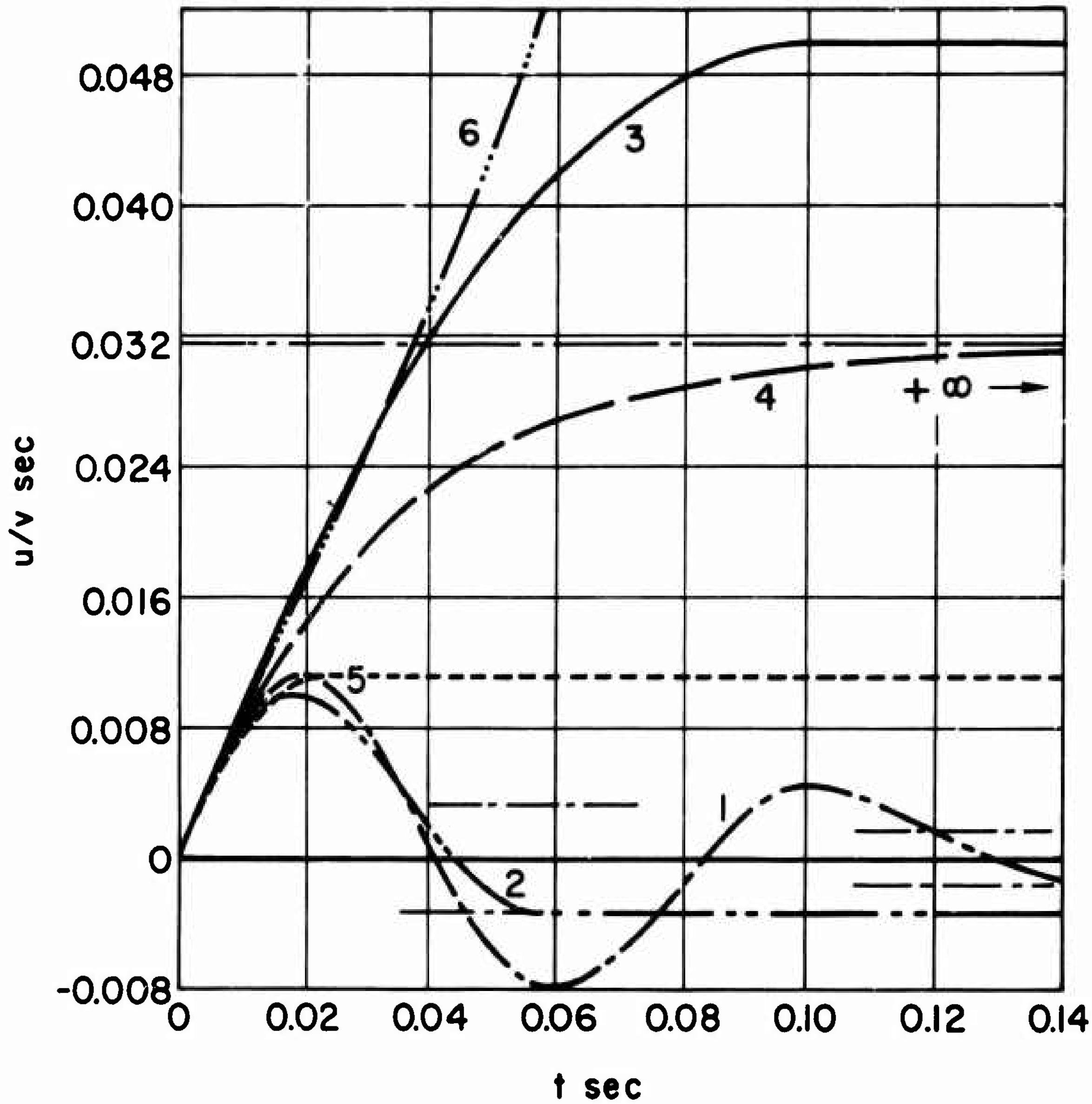

Fig. 12. Displacements of Model with Dry Friction (From Symposium of the Colston Research Society on Engineering Structures)

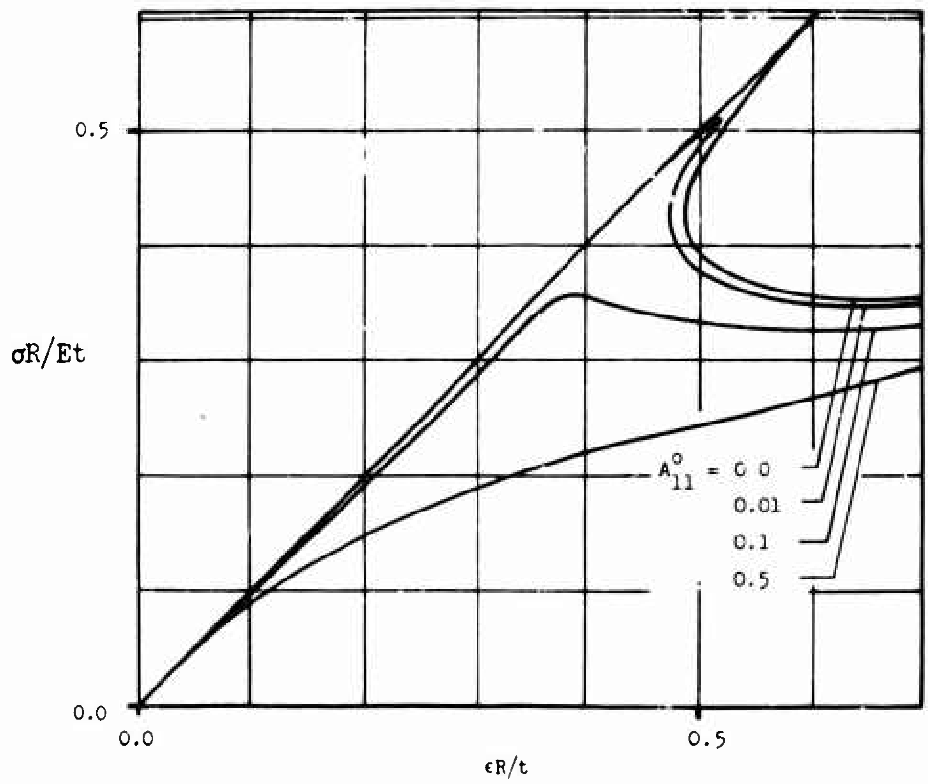

Fig. 13. Effect of Small Initial Deviations from Perfect Shape on the Maximum Load Carried by Circular Cylindrical Shell (From Report SUDAER 227 of Stanford University Department of Aeronautics and Astronautics)

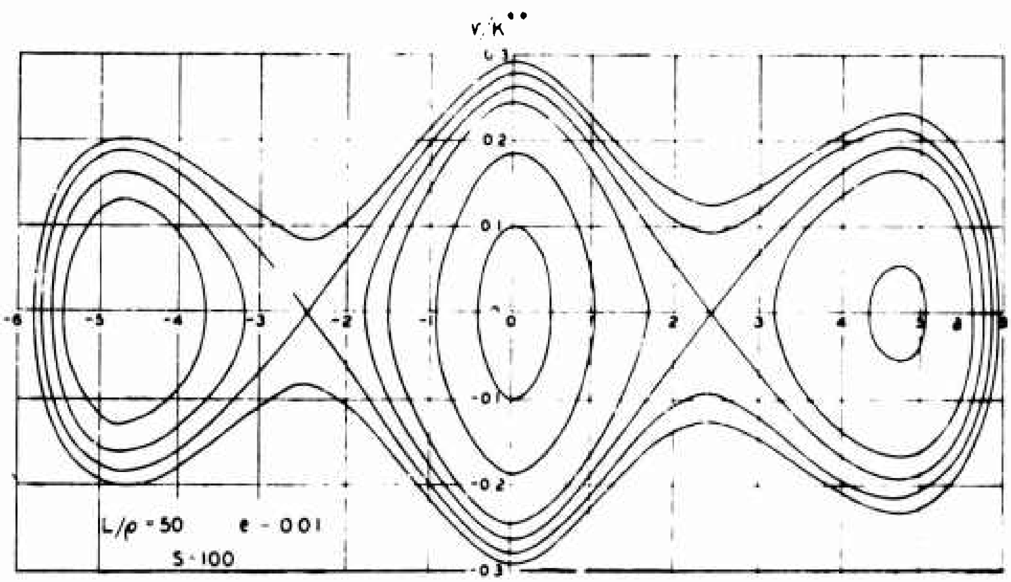

Fig. 14. Phase Plane Diagram of Oscillations of Imperfect Column in Elastic Testing Machine Near Maximum Load (From Journal of the Royal Aeronautical Society)

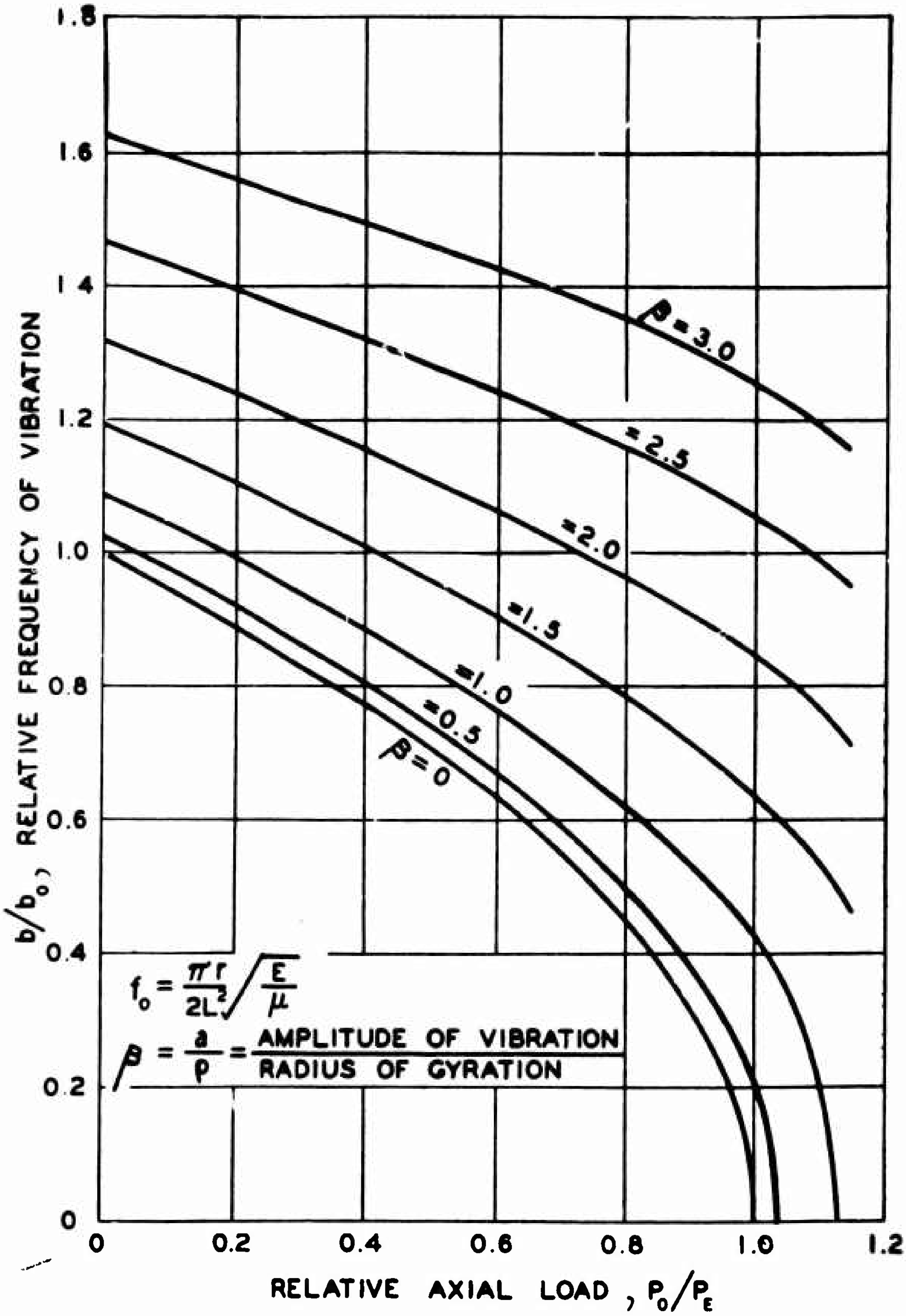

Fig. 15. Influence of Amplitude on Frequency of Oscillations of Column in Rigid Testing Machine (From Journal of Applied Mechanics)

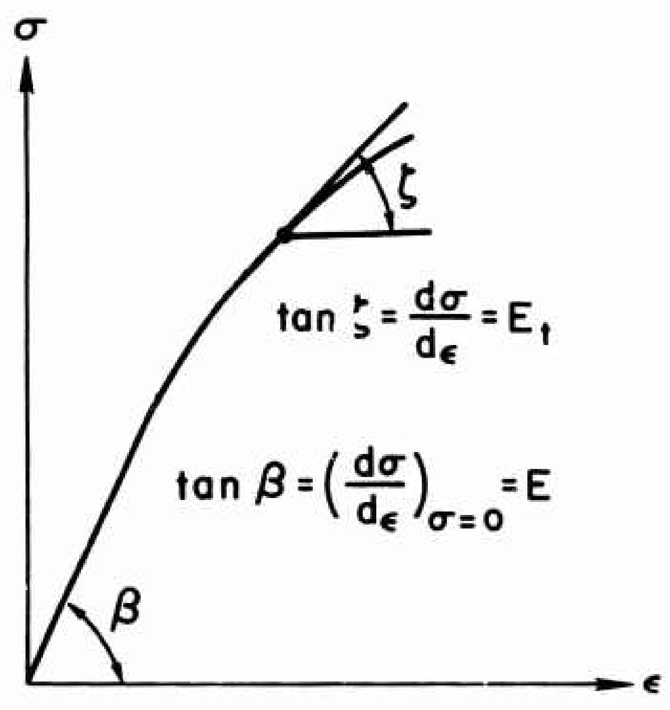

Fig. 16. Stress-Strain Curve and Tangent Modulus

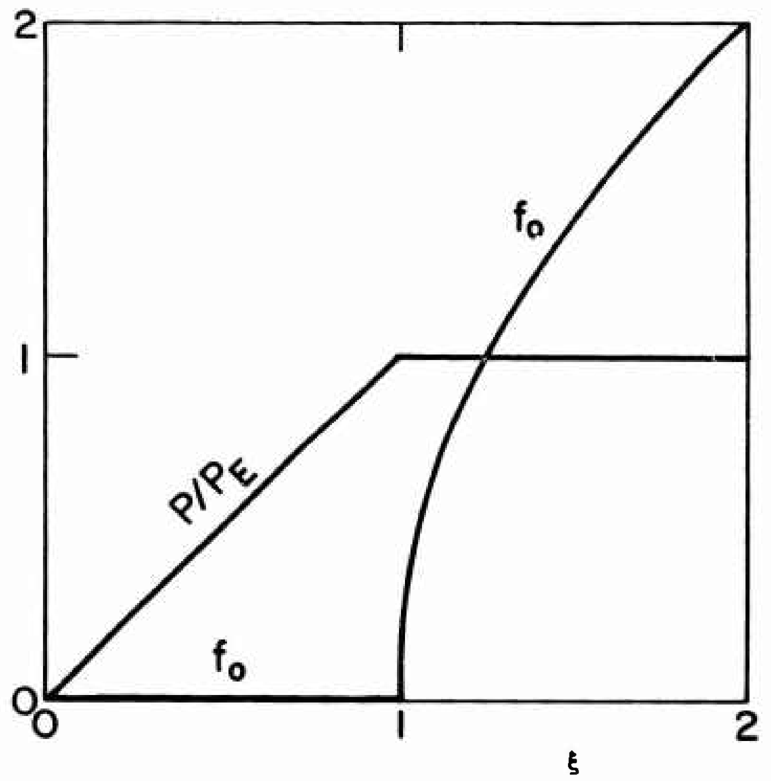

Fig. 17. Static Nondimensional Lateral Displacement Amplitude f_o and End Load Ratio P/P_E for Axially Compressed Perfect Column

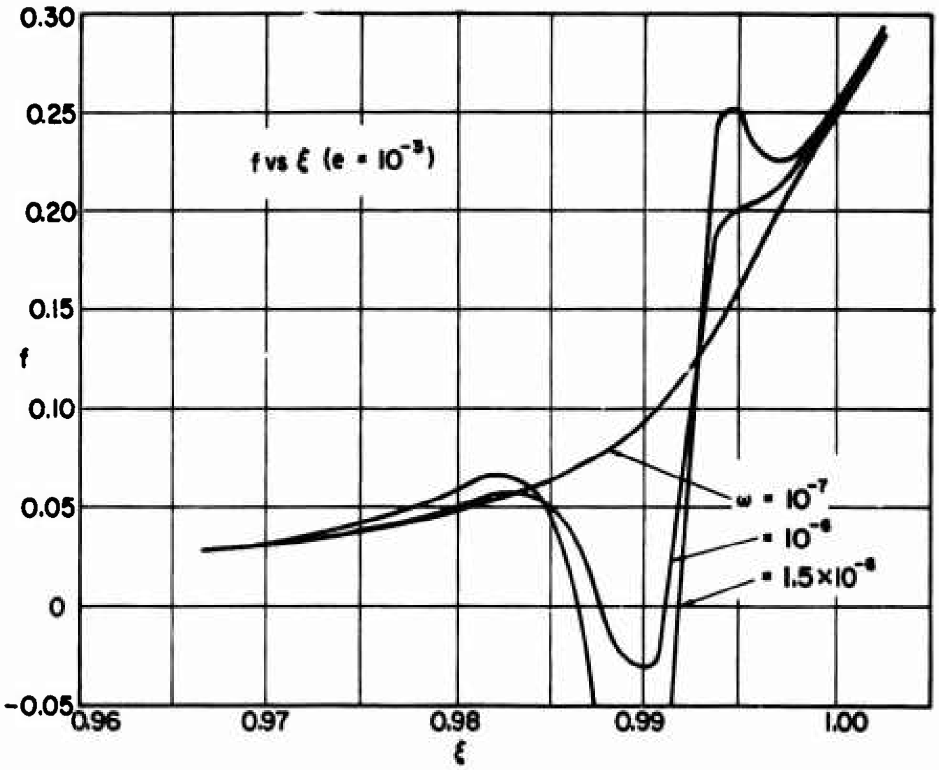

Fig. 18. Nondimensional Lateral Displacement Amplitude f versus Nondimensional Time ξ for Slowly Compressed Imperfect Column

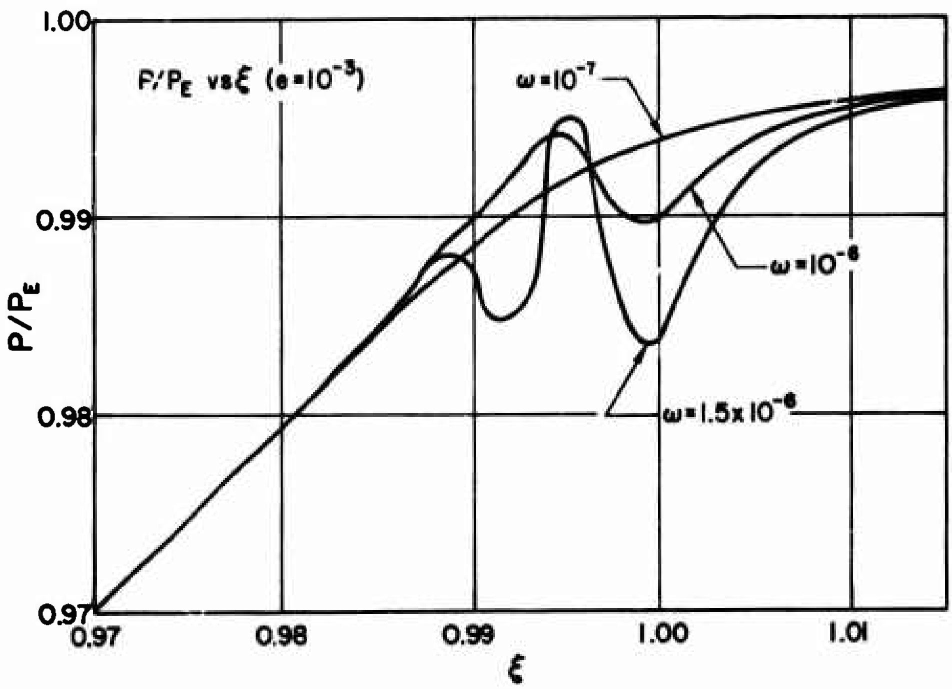

Fig. 19. Load Ratio P/P_E versus Nondimensional Time ξ for Slowly Compressed Imperfect Column

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Dynamic stability is defined and classified, and examples are given for the various classes of problems. Criteria are developed for practical stability and it is shown that in a practical elastic column tested in a conventional testing machine stress reversal always precedes the attainment of the maximum load. The two coincide, however, in the limit when the initial deviations of the column axis from straightness and the loading speed tend to zero.

14. KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
Thin shells						
Shell buckling						
Dynamic buckling						
Buckling in testing machine						
Arch						
Buckling of arch						
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